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Faculty of Electrical Engineering
Department of Cybernetics

Bachelor Thesis

Reconstruction of Spheres in 3D for Robot Calibration

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1. Study principles of sphere projection to uncalibrated and calibrated images [1,2].
2. Suggest an approach to reconstructing a pair of touching spheres in 3D space.
3. Implement the approach and verify it on real data.

Bibliography / sources:

[1] Hui Zhang, Kwan-Yee K. Wong, and Guoqiang Zhang. Camera Calibration from Images of Spheres. IEEE TRANSACTIONS ON PATTERN ANALYSIS AND MACHINE INTELLIGENCE, VOL. 29, NO. 3, MARCH 2007 (http://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=4069265)
[2] M. Agrawal and L. S. Davis, "Camera calibration using spheres: a semi-definite programming approach," Proceedings Ninth IEEE International Conference on Computer Vision, Nice, France, 2003, pp. 782-789 vol.2. doi: 10.1109/ICCV.2003.1238428

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I would like to thank my advisor Tomáš Pajdla for introducing me into the world of research, showing me how to creatively solve the problems in projective geometry and for his guidance and advices that helped me to finish this work.

Declaration

I declare that the presented work was developed independently and that I have listed all sources of information used within it in accordance with the methodical instructions for observing the ethical principles in the preparation of university theses.

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Abstract

Computer vision is a fast and widely developing branch of science. One of the problems and tasks of computer vision is the determining of a particular object on the image and its position in space. In this work, we propose our algorithm, which allows us to uniquely determine the sphere in space. We concentrate on studying the theory, which directly relates to the theme of this work and apply this theory in practice to obtain a solution.

As a result, we have a working algorithm that gives a satisfactory result on real data. In the future, this algorithm can be used, for example, to calibrate the robot in such a way that determining the position of the sphere will determine the position of the last joint of robotic arm.

Keywords: computer vision, conic sections, quadrics, camera calibration

x

Abstrakt

Po£íta£ové vid¥ní je rychle rozvijející se oblast v¥dy. Jedním z problem· a úkol· po£íta£ového vid¥ní je detekce konkretního objektu na obrázku a ur£ení jeho polohy v prostoru. V této práci navrhujeme algoritmus, který umoº¬uje jednozna£n¥ ur£it polohu sféry v prostoru. Soust°e¤ujeme se na teorii, která se p°ímo vztahuje k tématu této práce, a pouºiváme ji v praxi pro získání °e²ení.

Ve výsledku dostáváme funk£ní algoritmus, který dob°e funguje p°i zkou²ení na reálných datech. V budoucnu tento algoritmus m·ºe být pouºit, nap°íklad, pro kalibraci robota, a to tak, ºe ur£ení pozice sféry ur£í i pozici konce robotické ruky.

Klí£ová slova: po£íta£ové vid¥ní, kuºelose£ka, kvadrika, kalibrace kamery

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Chapter 1

Introduction

Computer vision is a fast and widely developing branch of science. In our time it is closely related to arti cial intelligence, namely, with the recognition and detection of objects. Extracting information from 2d images of 3D objects is also one of the most common branches of computer vision. It combines the solution of polynomial equations, linear algebra and projective geometry.

In this work, we focused our attention on recreating a sphere on the basis of its image. Our main task was to propose an algorithm that would allow unambiguous determination of the position of the sphere in space. Similar problem has been solved by Zisserman and Cross in [3]. They used two cameras to get stereo image and to determine the position of the object, called quadric.

The motivation for this work was a group of researchers who are engaged in calibrating the robot. They need to determine the position of the sphere in space, which is located on the end of the robotic arm, in order to accurately determine the coordinates of the last joint and to calibrate the robot.

This thesis can be divided to the following sections:

1. In the beginning we will conduct theoretical introduction in our problem, that is needed for better understanding the problem and that we needed to suggest the solution.
2. Then, we will move on with the main part of this thesis, namely we will suggest a solution to the given problem and discuss problems, that could show up.
3. After, we will test the solution on the real data and we will see the limitations of the algorithm.
4. Finally, we will summarize this work.

Chapter 2

Basic theory

In this chapter we will go through the theory, that we had to study to successfully tackle the issue. Theory consists of basic knowledge in computer vision and geometry, such as surfaces in 3D and di erent conic sections. We also studied the interaction between surfaces in 3D and its projections on the planes. Thus, we divide our theoretical introduction into the following sections

1. Perspective camera model
2. Conic sections
3. Quadrics
4. Projection of the sphere onto the image plane

2.1 Perspective camera model

There exist many types of cameras. We will concentrate ourselves on the perspective camera model, that maps points from 3D space onto the image plane. Basically, points on the image plane are given by the intersection between the vector, connecting the camera center with the point, and the image plane.

Chapters 6 and 7 in book [10] describe the basics of perspective camera model. In this section we will brie y summarize the main of these two chapters. We will use the notation of [10] and [5].

Let us consider following illustration 2.1

Figure 2.1: Coordinate systems of perspective camera [10].

Figure 2.1 shows the geometry of the perspective camera. We can see there the camera center C, the image plane and the point O, which is the origin for the main coordinate system.

To solve the problems of computer vision, such as extracting information about 3D object from 2D image, we will de ne some handful coordinate systems.

First, we introduce the world coordinate system $(O;)$, so we can express any point in the space with its help. This coordinate system consists of the origin O and orthonormal basis containing vectors $d_1; d_2$ and d_3, so every point can be written as

$$X = {}_1 d_1 + {}_2 d_2 + {}_3 d_3 \tag{2.1.1}$$

World coordinate system allows us to de ne the camera projection center C .

Next we de ne the image coordinate system $(o; \ )$ of the image plane . We use this coordinate system to express the points on the image plane through its origin o and two non-orthogonal basis vectors b_1 and b_2. So, any point on the image plane can be represented as vector in basis

$$u_i \ = u_i b_1 + v_i b_2 = \begin{matrix} u_i \\ v_i \end{matrix} \qquad (2.1.2)$$

Finally, to associate the coordinates of points on the image with the camera center, we introduce the camera coordinate system $(C; \ )$. Its origin is always placed in the projection center and its basis consists of basis with added vector b_3. We de ne vector b_3 as a vector connecting origins of two coordinate systems and . Thus, it is more convenient to write coordinates of the points in the image plane with respect to basis , because the third coordinate will always be equal to one. At this point, we can express above vector u in basis by simply adding one, i.e.

$$u_i \ = \begin{matrix} u_i \\ 1 \end{matrix} \qquad (2.1.3)$$

Now let us take an arbitrary point X in the space. If we start drawing the line from the camera center C to point X, we will reach the image plane at the point x. We de ne such point x as the projection of point X onto the image plane and we can represent it with respect to as

$$x \ = \begin{matrix} 2 \\ 4 \end{matrix} \begin{matrix} u \\ v \\ 1 \end{matrix} \begin{matrix} 3 \\ 5 \end{matrix} \qquad (2.1.4)$$

If we want to represent point X with respect to the camera coordinate system, we can imagine, that point X in can be obtained from x as its multiple by some non-zero number . Thus, we can write

$$x \ = X \quad C \qquad (2.1.5)$$

Now, suppose that we have the matrix A, that converts any vector from space, given with respect to the world coordinate system , to the vector represented by the camera coordinate system . We can write this as

$$x \ = Ax \qquad (2.1.6)$$

The matrix A, as we will see later, contains the *extrinsic* and *intrinsic* parameters of the camera. We will give the de nition to these parameters a little bit later.
Using Equation 2.1.6 we can re-write Equation 2.1.5 as

$$x \ = A(X \quad C) \qquad (2.1.7)$$

$$x \ = A \ \ I j \quad C \quad \begin{matrix} X \\ 1 \end{matrix} \qquad (2.1.8)$$

$$x = P \begin{matrix} X \\ 1 \end{matrix} \tag{2.1.9}$$

We just have introduced the 3 4 image projection matrix P .
With help of Equation 2.1.9 we can describe the relationship between the point in the space X and the point on the image planeu . If we know the value of and matrix A , we can obtain the coordinates of the point in space from Equation 2.1.7 as

$$X = A^{1}x + C \tag{2.1.10}$$

Now we will look closer to the matrix A . We de ne it as

$$A = \frac{1}{f}KR \tag{2.1.11}$$

where we introduce two new matrices K , R and parameter f , which is the focal length and is de ned as the distance between the camera center and the image plane.

We have already told, that the matrix A contains parameters of the camera. We have mentioned extrinsic and intrinsic parameters of the camera. Matrix K determines the intrinsic parameters, that are given by camera construction and do not change when moving or rotating the camera. This matrix is given as follows

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & 1 \end{bmatrix} \tag{2.1.12}$$

The elements of matrix K are de ned as

$$k_{11} = \frac{f}{kb_1k} \tag{2.1.13}$$

$$k_{12} = \frac{f\cos\angle(b_1;b_2)}{kb_1k\sin\angle(b_1;b_2)} \tag{2.1.14}$$

$$k_{22} = \frac{f}{kb_2k\sin\angle(b_1;b_2)} \tag{2.1.15}$$

Elements k_{13} and k_{23} determine the coordinates of the principal point in , that is given as intersection between vector c_3 and the image plane.

Now we will look at the second matrix introduced in Equation 2.1.11. This is 3 3 rotation matrix, that determines the extrinsic parameters of the camera, i.e. its orientation. Matrix R and vector C uniquely de ne the position and orientation of the camera.

To continue, we will de ne new orthogonal camera coordinate system (C;) with three basis vectors $c_1; c_2$ and c_3. These three vectors are chosen in such way, that Span$\{c_1; c_2\}$ de nes the plane parallel to the image plane . Vector c_3 is orthogonal to both c_1 and c_2 and thus is orthogonal to the image plane and has the length equals to the focal length.

The basis holds

$$c_1 = k_{11}b_1 \tag{2.1.16}$$

$$c_2 = k_{12}b_1 + k_{22}b_2 \tag{2.1.17}$$

$$c_3 = k_{13}b_1 + k_{23}b_2 + 1\,b_3 \tag{2.1.18}$$

By using matrices K, R, A and focal length we can provide next vectors transformations between coordinate systems

$$y = Ky \tag{2.1.19}$$

$$y = \frac{1}{f}Ry \tag{2.1.20}$$

$$y = Ay \tag{2.1.21}$$

Inverse transformations could be obtained by applying matrices K^{1}, R^{1} and A^{1}.

Using above equations we can express given vector with respect to any of the coordinate systems.

Using Equation 2.1.11 and Equation 2.1.8 we obtain

$$f\ x = KR\ \ Ij\ \ C\ \ \begin{matrix} X \\ 1 \end{matrix} \tag{2.1.22}$$

$$f\ x = P\ \begin{matrix} X \\ 1 \end{matrix} \tag{2.1.23}$$

where $3\ \ 3$ matrix P is called the camera projection matrix .

2.2 Conic sections

Due to the perspective camera model all projections of the objects do not preserve their actual size and in most cases may have di erent form, than in the real world. For example, if we project the circle, that does not lie on the plane, which is parallel to the image plane and its center does not intersect with the ray given by vector c_3, then we will obtain an ellipse. When increasing an angle between the image plane and the plane, where circles lie, the di erence between circle and ellipse would be more evident. Figure 2.2 clearly demonstrates this.

Figure 2.2: (a) The images of circles are almost represented as circles. (b) The circles are represented as ellipses [5].

When working with the images of spheres in real life, we are also dealing with circles and ellipses. By [4] conic section is the name given to the shapes that we obtain by taking di erent plane slices through a double cone. In our case the cone is produced by inscribed sphere and a vertex placed in the camera projection center. In general, we may consider the following conic sections

Figure 2.3: Di erent conic sections [4].

We can see, that there are 6 types of conic sections. However, in the real life the plane will never pass through the vertex, which follows from non-zero focal length. Also, we may consider nite image planes, which implies, that plane will never intersect given sphere. From above words follows, that the only conic section, which can be obtained is ellipse. To suggest the solution to our problem of reconstructing sphere in the space by given image we had to study basics of conic sections.

By [4] any conic has an equation of the form

$$Ax^2 + Bxy + Cy^2 + Fx + Gy + H = 0 \tag{2.2.1}$$

where A, B, C, F, G and H are real numbers, and not all of A, B and C are zero.
To work with conic sections in computer vision, we have to express them in the matrix form as

$$x^T A_{33} x + J^T x + H = 0 \tag{2.2.2}$$

where matrices A_{33} and J contain coe cients A, B, C, F, G and the vector x represents a point on the conic. These two matrices are given as

$$A_{33} = \begin{matrix} A & B=2 \\ B=2 & C \end{matrix} \tag{2.2.3}$$

$$J = \begin{matrix} F \\ G \end{matrix} \tag{2.2.4}$$

Equation 2.2.2 is the equation of the conic in inhomogeneous coordinates [5]. We can use this equation if we work only with coordinate system of the perspective camera model. To work with another coordinate systems, for example, we should use homogeneous coordinates.

So, we can re-write above equation as

$$X^T q X = 0 \tag{2.2.5}$$

where matrix q is the symmetric invertible (in case of ellipse, hyperbola and parabola) matrix and can be written as

$$q = \begin{bmatrix} A & B/2 & F/2 \\ B/2 & C & G/2 \\ F/2 & G/2 & H \end{bmatrix} \tag{2.2.6}$$

We can see, that the upper left 2×2 submatrix is the matrix A_{33}. The matrix A_{33} is very useful for us, because with its help we can determine if obtained conic section is a circle or an ellipse. If coe cients A and C are equal and $B = 0$, then conic section is a circle.

We have de ned conics through the points. Due to the duality we can represent the same conic through lines. We call this dual conic and it is also represented by 3×3 matrix q [5]. For dual conics we can write

$$l^T q\ l = 0 \tag{2.2.7}$$

where l is the tangent to a conic line. If matrix q is not singular and is symmetric, then $q = q^{-1}$. Later we will use this to obtain the projection of the sphere onto the image plane.

2.3 Quadrics

Spheres are wide-used objects in computer vision [6][7][9]. If given sphere without any markers on it, then such sphere does not have an orientation and thus is easier to describe. Also in many cases it is easy to detect the apparent contour of the sphere on the image. To work with spheres in computer vision, we would have to express them generally in matrix form.

First, we will write general equation for quadric surface [1]. Such surface is the set of points, that satisfy the equation

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Eyz + 2Fxz + 2Gx + 2Hy + 2Jz + K = 0 \tag{2.3.1}$$

The above equation can be represented in the matrix form by 4×1 homogeneous vector X, that represents a point on the quadric, and a 4×4 matrix Q as

$$X^T Q X = 0 \tag{2.3.2}$$

where matrix Q is

$$Q = \begin{bmatrix} A & D & F & G \\ D & B & E & H \\ F & E & C & J \\ G & H & J & K \end{bmatrix} \tag{2.3.3}$$

If we set $A = B = C = K = 1$ and $D = E = F = G = H = J = 0$, we get a unit sphere with center placed at the origin.

The sphere in general case is represented by its center $\mathbf{a}$ and radius r. If we de ne as $\mathbf{a}^T\mathbf{a} \quad r^2$ [6], sphere equation in matrix form will be represented as

$$Q = \begin{bmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_3 \\ a_1 & a_2 & a_3 & \end{bmatrix} = \begin{bmatrix} I & \mathbf{a} \\ \mathbf{a}^T & \end{bmatrix} \tag{2.3.4}$$

Obtained matrix Q will be used in the next section to construct a projection onto the image plane.

2.4 Projection of a sphere into a perspective image

In this section we will use knowledge obtained in the previous two sections to obtain conic section for a concrete given sphere.

Let us assume, that we are given the sphere

$$S \quad Q = \begin{bmatrix} I & \mathbf{a} \\ \mathbf{a}^T & \end{bmatrix} \tag{2.4.1}$$

and matrices K, R and a focal length. Thus, we can construct the camera projection matrix P. Now we will write the equation for the dual conic q [5][6] of conic q as

$$q \quad = PQ \quad P^T \tag{2.4.2}$$

where Q is the dual quadric of Q.

By substituting $q \quad = q \quad ^{1}$ and $Q \quad = Q \quad ^{1}$ into Equation 2.4.2 we get

$$q \quad ^{1} = PQ \quad ^{1}P^T \tag{2.4.3}$$

$$q = (PQ \quad ^{1}P^T) \quad ^{1} \tag{2.4.4}$$

Equation 2.4.4 is the projection of quadric Q onto the image plane.

Chapter 3

Sphere reconstruction

In this chapter we will suggest a solution to the problem of sphere reconstruction from the given image. Brie y, our idea is that conic section and the camera center de ne the concrete cone and there is only one point, where center of the sphere can be placed so that sphere would be inscribed in this cone.

When we want to recover the sphere from given conic section, we can obtain two cases:

1. Obtained conic section is an ellipse
2. Obtained conic section is a circle

3.1 Ellipse

Basically, we can divide our solution into the following steps:

1. Calculate eigenvectors and eigenvalues of the matrix A_{33}
2. Calculate the center and vertices of an ellipse
3. Find an angle between two vectors connecting major vertices and calculate a point M , where angle bisection and major axis intersect
4. Determine points X and Y where orthogonal to the major axis line passing through M intersects with ellipse.
5. Determine sphere's parameters by given radius and compute the distance between sphere's center and the camera projection center
6. Using obtained direction and distance, calculate the actual position of the sphere

Now we will go through each item above and look closer at the suggested solution.

3.1.1 Eigenvectors and eigenvalues of A_{33}

First, we need to extract the matrix q, representing the conic section. In the real scene we used "ellipse detection" for given photos and then more precisely found the contour of an ellipse and used program for ellipse tting [12].

Now assume we are given matrix q. We easily obtain matrix A_{33} from q and then calculate its eigenvectors and eigenvalues by the de nition

$$A_{33}e = ve \tag{3.1.1}$$

where e is the eigenvector corresponding to the eigenvalue v. Thus, we obtain two eigenvectors corresponding to two di erent eigenvalues, which implies that these vectors are orthogonal [8]. By the principal axis theorem, the axes of an ellipse are orthogonal, so they will be parallel to the eigenvectors of matrix A_{33}. Thus, by far we have two eigenvectors e_1 and e_2 and corresponding to them eigenvalues v_1 and v_2.

3.1.2 Ellipse's vertices

Our next step is to calculate the coordinates of the ellipse's center and then obtain the equations for the axes. We de ne the center of an ellipse as a point, where major and minor axes intersect. If we multiply rst two columns of matrix q by vector $x = \begin{matrix} x & y & 1 \end{matrix}^T$ we get two equations that give us one solution, which is our required center [2]. Thus, we will get

$$\begin{cases} Ax + (B=2)y + D=2 = 0 \\ (B=2)x + Cy + G=2 = 0 \end{cases} \tag{3.1.2}$$

By solving it we obtain the point $(x_c; y_c)$.

We have already seen, that the minor and major axes are parallel to two eigenvectors. The eigenvector corresponding to the smaller eigenvalue will be parallel to the major axis, and eigenvector corresponding to the larger eigenvalue will be parallel to the minor axis [2]. We need to nd two points, where major axis crosses an ellipse. Thus, we again solve the system of non-linear equations, which are

$$\begin{cases} x^T qx = 0 \\ y = \dfrac{e_2 x \quad x_c e_2 + y_c e_1}{e_1} \end{cases} \tag{3.1.3}$$

where x is $3 \quad 1$ vector representing homogeneous coordinates, q is the conic matrix and $(e_1; e_2)$ is the eigenvector corresponding to the smaller eigenvalue. Thus, we obtained two points in the image coordinate system , which can be represented in as vectors

$$a = \begin{matrix} a_1 & a_2 & 1 \end{matrix}^T \tag{3.1.4}$$

$$b = \begin{matrix} b_1 & b_2 & 1 \end{matrix}^T \tag{3.1.5}$$

3.1.3 Projection of the sphere's center

In the previous part we obtained two vectors a and b and now we will calculate the angle between them. We will use well known formula

$$\cos \; = \frac{a \quad b}{\|a\| \quad \|b\|} \tag{3.1.6}$$

Angle will be used a little bit later.

The second part of this step is to calculate the coordinates of point M and relevant vector m, given by intersection between bisection-line and the major axis. From geometry we know, that bisection will divide line into two parts that are proportional to the lengths of surrounding lines. This is illustrated in Figure 3.1

Figure 3.1: Angle bisection.

$$\frac{|BD|}{|BC|} = \frac{|AD|}{|AC|} \tag{3.1.7}$$

With its help we can calculate vector m as

$$m = \frac{\|a\|}{\|a\| + \|b\|} a + \frac{\|b\|}{\|a\| + \|b\|} b \tag{3.1.8}$$

Now we will deviate from our train of thought a bit and image the process of obtaining an ellipse as illustrated in Figure 3.2. Imagine, we have a cone and a plane, that intersects it, is parallel to the cone's base. Now we choose an axis around which we will rotate the plane . Our chosen axis u is the line, that lies in the plane and intersects with cone's height. Also, we denote two points, where u intersects with the cone, as X and Y. To achieve the rotation, we use some rotation matrix $R_u(\ )$. By the de nition of any rotation matrix we have

$$Rv = v \tag{3.1.9}$$

where v is vector that is parallel to the rotation axis. Equation 3.1.9 tells us, that after rotation vector v will be unchanged. We have a little bit more di cult situation, because we do not rotate the conic section, but we want to rotate the plane , so we could get new conic sections. In our case the only unchanged line in every obtained ellipse (and a circle, which we had at the beginning) after application the rotation, is the line that coincides with

axis u. Such line will maintain its length and orientation, unlike major and minor axes of obtained ellipse.
This approval can be demonstrated in the following illustrations in Figure 3.2

Figure 3.2: Line lying on the rotation axis does not change the length [11].

We see, that lengths X^0Y^0 and $X^{00}Y^{00}$ are equal. Due to the fact, that cone is constructed by given sphere and vertex in the camera center we can tell, that the cone's height will pass through the sphere's center and if we take any two opposite rays, that have beginning in the vertex and if these to rays and the cone's height form a plane, then the cone's height will be the bisection of the angel between two rays. In the set of such pairs of rays only two rays can easily determine an angle. We can see in Figure 3.2, that such rays will pass through major vertices of obtained conic section. From all of the above we can conclude, that some multiple of the vector obtained in Equation 3.1.8 will pass through the sphere's center.

3.1.4 Rotation axis

From the above part we can see how useful are points, where orthogonal to the major axis line, which also contains point M, intersects with an ellipse. We have already obtained the equation of the line, that is the major axis, by solving system of equations 3.1.3. We also know, that this line was given by the point $(x_c; y_c)$ and direction vector $\begin{matrix} e_1 & e_2 \end{matrix}^T$. The line, that is orthogonal to the major vertex will have the direction vector equals to $\begin{matrix} e_2 & e_1 \end{matrix}^T$.
If we assume, that the point M has coordinates $(m_1; m_2)$, we can obtain the intersection

between the line and conic by solving next system of non-linear equations

$$\begin{cases} x^T q x = 0 \\ y = \dfrac{e_1 x - e_1 m_1 - e_2 m_2}{e_2} \end{cases} \tag{3.1.10}$$

The solution will give us two points X and Y, which, as we saw in Figure 3.2, correspond to X^0 and Y^0 (same as X^{00} and Y^{00}). Now we will compute the length l of XY by very well known formula

$$l = \sqrt{(X_1 - Y_1)^2 + (X_2 - Y_2)^2} \tag{3.1.11}$$

We will use obtained length l in the next step.

3.1.5 Sphere's parameters

In this part of solution we need to switch to 2D to easily calculate the distance between the camera center and the sphere's center.

If we build a plane, that passes through the cone's vertex and line XY, we will get the following cross section

Figure 3.3: 2D view of sphere's recovering problem.

The sphere inscribed in the cone is reduced to the circle inscribed in the triangle with radius r. Our goal in this step is to calculate the distance CS, which we define as d. Further, we define the diameter of the visible contour of the sphere, that is circle with diameter

$EF = d_c$. The distance to the sphere's center will be a sum of $SL = n$ and $CL = p$. Both lengths we can compute as

$$n = r \sin\frac{}{2} \tag{3.1.12}$$

$$p = \frac{km\,kd_c}{l} \tag{3.1.13}$$

where $d_c = 2r\cos\frac{}{2}$. Then the distance between the camera center and a sphere is given by $d = (n + p)$. In the next step we will nally determine the position of the sphere in the space.

3.1.6 Recovering the position

The sphere's center is given by vector s, that has the direction and magnitude. We have already obtained the direction of this vector in . We need to normalize vector m and multiply it by distance d. Then we have to transform obtained vector into coordinate system. It could be done as follows

$$s = A^{1}\frac{m}{km\,k}d = f\,R^{1}K^{1}s \tag{3.1.14}$$

Now we have to add vector C to compute the actual sphere's center. Thus, we have

$$X_s = C + s \tag{3.1.15}$$

where X_s is the vector, determining the position of the sphere.

3.2 Circle

The second case can be achieved only if the sphere's center lies along c_3 of coordinate system. Unlike rst case, we do not have to calculate the direction, because it is already given by c_3. All we need to do is to calculate the distance from the camera center to the sphere's center. Methods for circle detecting in the images return the circle's center and its radius, so we can use Equations 3.1.12, 3.1.13, 3.1.14 and 3.1.15 to obtain the sphere's center.