Czech Technical University in Prague
Faculty of Electrical Engineering

DOCTORAL THESIS

BY

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Czech Technical University in Prague
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Department of Control Engineering

SPATIALLY INVARIANT SYSTEMS: MODELLING, ANALYSIS AND CONTROL VIA POLYNOMIAL APPROACH

Doctoral thesis
by

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Ph.D. programme:	Electrical Engineering and Information Technology
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Contents

CHAPTER 1

Introduction

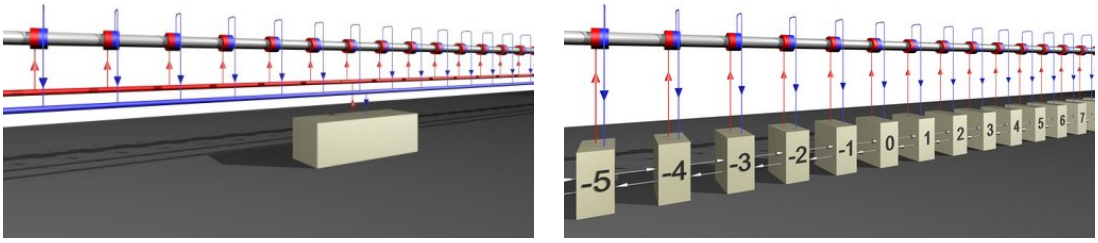
This thesis deals with modelling, analysis and control of a special subclass of spatially distributed systems, namely, linear time-invariant and spatially invariant systems. These systems are spatially discrete (discretised) because they are supported by a regular array of sensors and actuators. This thesis uses fractions of multivariate polynomials for modelling such systems, hence, algebraic operations with multivariate polynomials are the major tools both for analysis and synthesis.

1.1. SPATIALLY INVARIANT SYSTEMS

Spatially distributed systems are a class of *multidimensional* systems. They evolve in more than one dimension. This is the main difference from the *systems with lumped parameters* (also called *one-dimensional* systems), which evolve in only one dimension, corresponding to time. In this thesis, we consider spatially distributed systems with both temporal and spatial variable, which are *linear*, *time* and *spatially invariant*. Such systems can be mathematically described by linear *partial differential equations* (PDEs) with the time variable and variable(s) corresponding to space and with constant coefficients. An example is a heat conduction in a rod, depicted in Fig. 1.1. A temperature of a rod evolves in time as well as in space. Another examples include flexible structures, deformable mirrors, signal processing and filtering, vehicle platoons, etc.

1.2. CONTROL OF SPATIALLY DISTRIBUTED SYSTEMS

Control of spatially distributed systems has always been a very active research topic with engineering applications in many areas. An amount of research has been con-



(a) (b)

Fig. 1.1. Difference between centralised and distributed control of spatially distributed systems shown on an example of temperature regulation in a rod. (a) Centralised control of a distributed parameter system: a rod with an array of heaters and temperature sensors and one controller. (b) Distributed control of a distributed parameter system: a rod with an array of heaters and temperature sensors and a distributed controller (an array of controllers).

ducted on control design for PDEs. One group of such design methods relies on the possibility to affect the behaviour of the system (that is, a solution in the domain) by controlling the boundary conditions, so-called *boundary control*. However, this thesis focuses on systems featuring a regular array of sensors and actuators stretching all over the domain. This grid then enforces a spatial discretisation of the system.

Provided the parameters of the system do not depend on location, the resulting mathematical model is *shift invariant*. This assumes the domain is infinite, that is, the boundaries are at infinite distance, which is not realistic. Nonetheless, the assumption of shift invariance seems to be a reasonable simplification for design. Moreover, it is assumed that exciting the system at any location, response can only be observed in the neighbouring nodes of the sensor network, in other words, the dynamics of the system is *localised*. In addition, it makes possible to design and implement a distributed (decentralised) controllers with a regular mesh of sensors and actuators.

The basic difference between centralised and distributed control of spatially distributed systems is illustrated in Fig. 1.1. The design and implementation of the lumped (centralised) control scheme is feasible, if a particular application requires hundreds of control inputs and measured outputs. However, some of the relevant applications now emerging will require thousands of actuators and sensors. For such cases the central approach is inefficient and the only feasible approach appears to be the design and implementation of decentralised controllers with a regular mesh of sensors and actuators.

1.3. OBJECTIVES OF THE THESIS

The objective of the thesis is to develop practical methodology for control design for spatially distributed systems via polynomial techniques. In particular, to

- give a method to model a spatially invariant system by transfer function,
- establish necessary and sufficient condition for stability and develop a simple method to check whether a system is stable or not,
- find calculus to design a stabilising controller and a controller optimal in the sense of minimising the classical quadratic criterion.

1.4. OUTLINE OF THE THESIS

The thesis is organised as follows.

- Chapter 2 provides a short review of existing interesting and useful approaches to problems of control of spatially distributed systems.
- A short introduction to partial differential equations and mathematical description of spatially distributed systems is given in Chapter 3. Finite difference methods and explicit and implicit difference schemata are briefly explained. Examples of spatially distributed systems are given.

  Finite difference methods are used for modelling of spatially distributed systems. A model discrete in both time and space and a model continuous in time and discrete in space are derived. By means of an example, model of a heat conduction in a rod is obtained.
- Chapter 4 deals with modelling by a transfer function.
- Positive polynomial approach to stability analysis is given in Chapter 5.
- Chapter 6 proposes some control design techniques for spatially distributed systems based on operations with polynomials. A method of stabilisation is based on the semidefinite programming. The LQG controller is designed using multivariate polynomial spectral factorisation. A parametrisation of all stabilising controllers is established and a controller optimal in the sense of a criterion is selected. The dead-beat control is also mentioned.
- Chapter 7 brings a scope of tools which the methods presented in Chap. 6 need from a computing point of view.
- Finally, Chapter 8 is a comparison of methods given in this thesis to existing approaches. A controller is designed using a simple method of theory of sys-

tems with lumped parameters. A method for control of spatially distributed systems based on another principle than this thesis deals with is also used.

1.5. ORIGINAL RESULTS AND CONTRIBUTIONS

The thesis brings some results that are believed to be original. In Chap. 5, a method for stability analysis of a spatially distributed system was developed based on an equivalence of stability of a 2-D polynomial and positiveness of a certain symmetric polynomial (Schur-Cohn, Hermite-Fujiwara) matrix on the unit circle, and semidefinite programming formulation.

This approach is extended to a controller design in Secs. 6.1 and 6.2, where a method of stabilisation is proposed. In the discrete-time case, a factorisation of the Schur-Cohn matrix is used. If the order of the closed-loop system in the time variable is limited to be less than or equal to 1, a function of the coefficients of characteristic polynomial can be brought in as a criterion and the controller can be optimised. The degree in the space variable can be arbitrary. If the order in the time variable is greater than 1, the method is able to stabilise a system, however, the possibility to optimise a controller is lost. In the continuous-time case, an analogous factorisation of the Hermite-Fujiwara matrix is unknown. The order of the closed-loop system in the time variable is limited to be less than or equal to 2. Well-known conditions for stability of a polynomial of the degree 2 can be used. A controller can be optimised like in the discrete-time case.

Sec. 6.3 proposes an extension of the LQG control to here considered class of spatially distributed systems. Theorem is proved. The $\mathcal{H}_2$-optimal control and a dead-beat control technique is proposed in Secs. 6.4 and 6.5. Methods presented here were derived as extensions of techniques well-known in 1-D systems.

1.6. PUBLICATIONS RELATED TO THE THESIS

The following papers were prepared within the thesis.

Journals articles

P1. Cichy, B. – Augusta, P. – Rogers, E. – Gałkowski, K. – Hurák, Z. On the control of distributed parameter systems using a multidimensional systems setting. *Mechanical Systems and Signal Processing*, 22(7):1566–1581. 2008.

P2. Cichy, B. – Augusta, P. – Rogers, E. – Gałkowski, K. – Hurák, Z. Robust control of distributed parameter mechanical systems using a multidimensional systems approach. *Bulletin of the Polish Academy of Sciences*, 58(1):67–75. 2010.

P3. Augusta, P. – Hurák, Z. Distributed stabilization of spatially invariant systems: positive polynomial approach. Submitted to *Multidimensional systems and signal processing*. December 2009.

Conference papers

P4. Augusta, P. – Hurák, Z. *Polmat: MuPAD library for symbolic computation with polynomial matrices*. Preprints of the 16th World Congress of the International Federation of Automatic Control [CD-ROM]. 2005.

P5. Augusta, P. – Hurák, Z. *Multidimensional transfer function model of a deformable mirror in adaptive optics*. Proceedings 17th International Symposium on: Mathematical Theory of Networks and Systems [CD-ROM]. 2006.

P6. Augusta, P. – Hurák, Z. – Rogers, E. *An algebraic approach to the control of spatially distributed systems: (2+1)D case with a deformable mirror example*. Proceedings of the 2007 IEEE International Workshop on Multidimensional (nD) Systems [CD-ROM]. Aveiro: University of Aveiro, 2007.

P7. Augusta, P. – Hurák, Z. – Rogers, E. *An algebraic approach to the control of spatially distributed systems — the 2-D systems case with a physical application*. Proceedings of IFAC Sympsium on Systems, Structure and Control 2007 [CD-ROM], 2007.

P8. Augusta, P. – Hurák, Z. *Algebraic approach to LQ-optimal control of spatially distsributed systems: 2-D case*. Preprints of the 17th IFAC World Congress. 2008.

P9. Augusta, P. – Hurák, Z. *POLMAT library now within Symbolic Math Toolbox for Matlab in multidimensional systems computations*. Proceedings of the 6th International Workshop on Multidimensional (nD) Systems, Eds: Karampetakis, N. – Galkowski, K. – Rogers, E. 2009.

P10. Augusta, P. – Hurák, Z. *Distributed stabilization of spatially invariant systems: positive polynomial approach*. Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems. 2010.

1.7. ACKNOWLEDGEMENTS

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CHAPTER 2

State of the art

It turned out as early as in the late 1960s and early 1970s that spatially distributed systems can be studied within a broader class of systems whose coefficients are functions of parameters. The right mathematical concept appeared to be that of linear systems over rings, because the coefficients in the state-space matrices and the coefficients in the transfer functions are elements of a ring. This broader class of systems also includes systems with delays or systems over integers. Among the pioneers in the area of linear systems over (commutative) rings were Kalman and his doctoral student Rouchaleau [57] and Kamen [35]. Readable surveys were given by Sontag [64] and Kamen [36]. Specialisation of these general results to spatially distributed systems was given in another survey [10] by Kamen.

A few papers followed in the early 1980s such as [37] and [38], but the interest of the community into this field faded away towards the end of 1980s and throughout 1990s. Surprisingly, the field was revived around the beginning of the new millennium, through the papers [8, 25, 39, 13, 24, 66, 32, 65, 23], just to name a few. A related field of repetitive systems [56] also comes along with this revival. Another approach to distributed control can consist in developing efficient algorithms for matrices with special structure, so-called a structure matrix approach. Using such special algorithms a huge system can be described and controlled by a method of centralised control. One of works on this topic is [55].

Let us have a look at the most relevant works in more detail.

Kamen [10, Chap. 4] proposes quite comprehensive methodology for analysis and synthesis of spatially distributed systems. He introduces the state-space representation with matrices depending on the variable corresponding to space shift. He introduces the transfer function description as a fraction of multivariate (n-D) polynomial matrices. After these definitions, he discusses basic properties like internal and input-output stability, reachability and stabilizability. The problem of stabilizability is studied also in terms of the Riccati equation. Solving the Riccati equation with matrices depending on the variable corresponding to space shift is considered.

Stabilisation by dynamic output feedback is proposed. The ideas are shown on an example of a seismic cable used in offshore oil exploration.

The Riccati equation with non-constant matrices is also solved in [48], where the optimal control of the string vehicles is studied. The system is described by state-space constant matrices, but space-shift variable arises in the weight matrices in, apart from that, standard quadratic criterion.

Also in [8], a system is described by state-space equations with matrices depending on the variable corresponding to space shift. Then the Lyapunov and the Riccati equation are proposed to solve and criteria for distributed LQ controller and $\mathcal{H}_2$- and $\mathcal{H}_\infty$- optimal controllers are defined like in theory of systems with lumped parameters with the modification that constant matrices become non-constant ones.

The papers [39, 13, 25, 23, 65] deal with modelling and control of systems with more than one spatial variable. A plate deflection is often used for illustration.

In [39], finite differences methods and the rectangular grid are used to discretise a partial differential equation describing a plate. The obtained model is continuous in time and discrete in space. It is ordinary differential equation, where space-shift operators appear.

The paper [13] also focuses on systems whose model is spatially discrete. Tools for analysis, synthesis and implementation of distributed controllers are developed and stated in terms of linear matrix inequalities.

In [25], a plate is modelled. However, unlike [39], a hexagonal grid is here used for spatial discretisation. Then, the model in the form of the transfer function is derived. The actuator command is assumed as an input u while the deflection of the plate is an output y. Moreover, the transfer function is considered to be *separable* and it is written in the form of product of two transfer functions

$$y = G(\lambda_1, \lambda_2)\, g(z)\, u,$$

where G describes deformation of the plate and depends only on variable corresponding to unit shift along the spatial axes λ_1, λ_2 and g describes the actuators dynamics which is considered to be the same for all actuators. The g depends only on the unit delay operator z.

In [23], a spatially distributed system is considered to be *spatially symmetric*. A separable model of a plant is used as it was introduced in the previous paper. A controller is assumed *spatially localised*. In other words, the control signal is computed based on only a finite number of error and actuator signals at nearby actuator cells. The goal of control law is to cancel the steady-state error in reaching the desired spatial profile.

In [65], a control with fixed structure is designed. The design is reduced to choosing parameters of the structure that closed-loop design objectives are satisfied. Furthermore, the authors of the paper take into account the fact that design objectives achieve their extremes at temporal steady state. The temporal degree is removed for the design process. The design objectives are expressed in the form of linear inequalities with design parameters. This technique is illustrated using an example of control of large flexible reflector. The reflector is divided into many cells with hexagonal arrangement. So-called “one-ring” controller, “two-ring” controller, etc. are defined and designed. “One-ring”, “two-ring”, ... mean that control law for each cell uses only its own local measurement plus measurement from one, two, ... rings of cells immediately surrounding it, respectively.

The papers [10, 48, 8, 13] propose quite comprehensive methodology, based on the state-space description of a plant. The papers [65, 23] deal with the transfer-function description, however, the goal of control strategy is to cancel the steady-state error and the temporal degree is removed. In this thesis, the author tries to give the methodology for analysis and synthesis of spatially invariant systems with the temporal variable based on the transfer-function description.

CHAPTER 3

Modelling of spatially distributed systems

This chapter provides a very short introduction to modelling of spatially distributed systems. In the first part, a classification and examples of partial differential equations (PDEs) are given and using finite difference methods for discretisation of PDE is discussed. The second part of chapter shows by means of an example a step-by-step derivation of a discrete in space model of a PDE. There is a lot of literature on these topics, an enquiring reader is referred to e. g. [50, 67, 7, 54, 63].

3.1. CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

Spatially distributed systems are mathematically described by PDEs. The most common types of PDEs are called *elliptic*, *parabolic* and *hyperbolic* equations. In what follows we roughly clear these terms on examples of second-order PDEs. Let $a > 0, b > 0, c > 0$ be constants, u be a solution, f right-hand side, ∇ denotes del and $\Delta = \nabla^2$.

The elliptic equation has the form

$$-c\,\Delta u - b\,\nabla u - a\,u = f.$$

Elliptic equations do not contain the time variable. They describe stationary states whose control is not subject of this thesis.

The parabolic equation has the form

$$d\frac{\partial u}{\partial t} - c\,\Delta u - b\,\nabla u - a\,u = f, \tag{3.1}$$

where $d > 0$. Parabolic equations contain first derivation with respect to time. A heat conduction, diffusion, chemical reactions and others irreversible processes are described by a parabolic PDE. If boundary conditions are non-negative and f is non-negative, then a solution u is non-negative too.

The hyperbolic equation has the form

$$d_1 \frac{\partial^2 u}{\partial t^2} + d_2 \frac{\partial u}{\partial t} - c\,\Delta u - b\,\nabla u - a\,u = f,$$

where $d_1 > 0$ and $d_2 > 0$. Many of the equations of mechanics, including waves, oscillations and deformations, are hyperbolic.

3.1. Example. A typical representative of parabolic PDE is *a heat conduction*. A model of heat conduction in a rod equipped with an array of temperature sensors and heaters is schematically sketched in Fig. 3.1. It is described by well-known heat equation

$$\frac{\partial u(x,t)}{\partial t} = \kappa \frac{\partial^2 u(x,t)}{\partial x^2} + f(t,x), \tag{3.2}$$

where u denotes temperature (°C), f the input heat ($^\circ\mathrm{C\,s^{-1}}$), t and x denote time (s) and a spatial coordinate (m), respectively, and $\kappa = \frac{\varkappa}{\rho\, c_p}$ is a constant ($\mathrm{m^2\,s^{-1}}$), where $\varkappa$ is the thermal conductivity ($\mathrm{W\,m^{-1}\,K^{-1}}$), ρ is the density ($\mathrm{kg\,m^{-3}}$) and c_p is the heat capacity per unit mass ($\mathrm{J\,K^{-1}\,kg^{-1}}$). □

3.2. Example. As a more difficult system *a thin flexible plate* can be considered. Its dynamics is described by hyperbolic equations [70]

$$\frac{\partial^4 w(x,y,t)}{\partial x^4} + 2\,\frac{\partial^4 w(x,y,t)}{\partial x^2\,\partial y^2} + \frac{\partial^4 w(x,y,t)}{\partial y^4} + \frac{\rho}{D}\,\frac{\partial^2 w(x,y,t)}{\partial t^2} = \frac{f(x,y,t)}{D}, \tag{3.3}$$

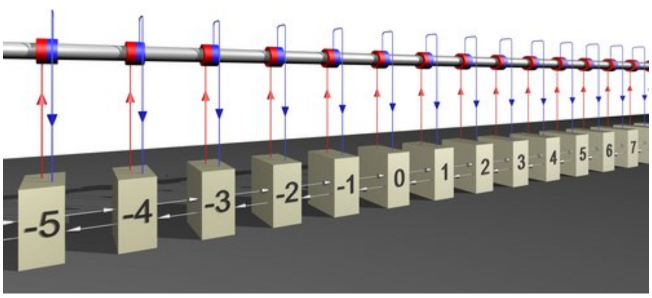

Fig. 3.1. A rod with an array of heaters and temperature sensors

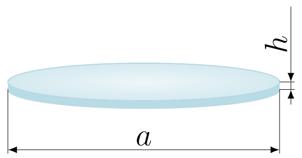

Fig. 3.2. Thin circular plate

where w is the lateral deflection in the z direction [m], ρ the mass density per unit area [kg/m^2], f the transverse external force, with dimension of force per unit area [N/m^2], $D = E\,h^3/(12\,(1-\nu^2))$, ν is Poisson ratio [], h is thickness of the plate [m] and E is Young's Modulus [N/m^2]. The flexible plate is sketched in Fig. 3.2.

3.2. FINITE DIFFERENCE METHODS

The aim is to base the control on action performed by an array of actuators and sensors. The presence of such array naturally implies the discretisation of PDE in spatial variables. In case of digital implementation of control, the PDE must be discretised with respect to time as well. We use *finite difference methods* (FDM) to proceed this.

FDM are a very popular numerical tool for solving PDEs, see e. g. [67, 63, 50, 54, 7]. The basic principle is to cover the region where a solution is sought by a regular grid and to replace derivatives by differences using only values at these nodal points. There are many types of grids which can be used, for example, rectangular, hexagonal, triangular or polar, applicable namely to PDEs in more then one spatial variable.

As an example [50] consider PDE

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x). \tag{3.4}$$

We seek approximation of solution at nodes

$$x_j = j\,\Delta x, \quad t_n = n\,\Delta t, \qquad j = 0,1,\ldots,J, \quad n = 0,1,\ldots,$$

where $\Delta x = 1/J$.

To solve (3.4) by FDM we shall replace the derivatives by finite difference and then solve the resulting difference equation in an evolutionary manner starting from $n = 0$. The simplest difference scheme uses a forward difference for time derivative

$$\frac{u(x_j,t_{n+1}) - u(x_j,t_n)}{\Delta t} \approx \frac{\partial u}{\partial t}(x_j,t_n)$$

and central second difference for the second order space derivative

$$\frac{u(x_{j+1},t_n) - 2\,u(x_j,t_n) + u(x_{j-1},t_n)}{(\Delta x)^2} \approx \frac{\partial^2 u}{\partial x^2}(x_j,t_n).$$

Equating the left-hand sides of the above two formulae gets an approximation of (3.4)

$$u(x_j,t_{n+1}) = u(x_j,t_n) + \mu\big[u(x_{j+1},t_n) - 2\,u(x_j,t_n) + u(x_{j-1},t_n)\big], \tag{3.5}$$

where $\mu = \Delta t/(\Delta x)^2$. Fig. 3.3 shows points used in the above schema. Each value

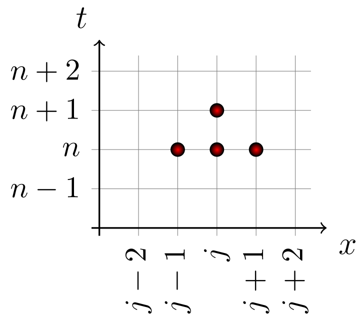

Fig. 3.3. An explicit difference scheme

at time level t_{n+1} can be independently calculated from values at time level t_n. For this reason (3.5) is called an *explicit difference scheme*. As you can see in [50], calculations using (3.5) show that numerical results depend critically on the value of μ. Derivation of explicit difference scheme has to be followed by analysis of convergence to a solution, i. e. finding relation between Δt and Δx which satisfaction will guarantee the convergence, see Sec. 3.4. In other words, explicit scheme has stability limit, which should be serve restriction and implies that very many time steps will be necessary to follow the solution over a reasonably large time interval.

Use of backward time difference gives a difference scheme which avoids this restriction. Replacement the forward time difference by the backward time difference in (3.5) gives *implicit difference scheme*

$$u(x_j,t_n) = u(x_j,t_{n-1}) + \mu\big[u(x_{j+1},t_n) - 2\,u(x_j,t_n) + u(x_{j-1},t_n)\big], \tag{3.6}$$

where $\mu = \Delta t/(\Delta x)^2$. Fig. 3.4 shows points used in (3.6). This schema contains three unknown values of u on the new time level n. In other words, we are not able to calculate the value $u(x_j,t_n)$ because its two neighbouring values in the equation, $u(x_{j-1},t_n)$ and $u(x_{j+1},t_n)$, are also unknown. This leads to solving a system of equations to get the values simultaneously. The importance of implicit method lies in that there is no any stability restriction on Δt. See [50] for proof and more details.

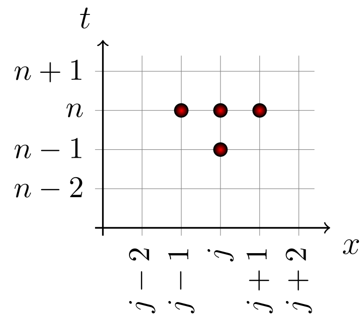

Fig. 3.4. An implicit difference scheme

3.3. DISCRETISATION OF PARTIAL DIFFERENTIAL EQUATION

By means of an example, we will use FDM to derive a description of a system in the form of a partial recurrence equation (PRE), discrete in both time and space. Consider the system of heat conduction in a rod[1] of Fig. 3.1 described by the parabolic PDE (3.2). Let T denote the sampling (time) period and h the distance between the nodes along the rod. Express the derivatives in (3.2) as differences corresponding to the grid

$$\left(\frac{\partial u}{\partial t}\right)_{k,i} = \frac{u_{k+1,i} - u_{k,i}}{T}, \qquad \left(\frac{\partial^2 u}{\partial x^2}\right)_{k,i} = \frac{u_{k,i-1} - 2\,u_{k,i} + u_{k,i+1}}{h^2},$$

where k corresponds to discrete time and i to the coordinate of the node. Substitution of the above formulae into (3.2) gives PRE

$$u_{k+1,i} = \frac{T\kappa}{h^2} u_{k,i-1} + \left(1 - 2\frac{T\kappa}{h^2}\right) u_{k,i} + \frac{T\kappa}{h^2} u_{k,i+1} + q_{k,i}, \tag{3.7}$$

where, for brevity, $q_{k,i} = T\, f_{k,i}$.

In the case we aim to derive a continuous-time model, we use a spatial discretisation scheme in the form

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i} = \frac{u(t)_{i-1} - 2\,u(t)_i + u(t)_{i+1}}{h^2}$$

and get

$$\left(\frac{\partial u(t)}{\partial t}\right)_{i} = \kappa\frac{u(t)_{i-1} - 2\,u(t)_i + u(t)_{i+1}}{h^2} + f(t)_i. \tag{3.8}$$

[1] The thin flexible plate described by (3.3) was modelled by author of this thesis in [2, 12].

3.4. ANALYSIS OF STABILITY OF THE DISCRETISATION SCHEME

To find out whether the spatially and temporarily discretised model gives an acceptably precise approximation to the original model described by PDE, to perform *von Neumann's analysis* of stability of an iterative finite difference scheme is necessary. The objective of this analysis is to determine whether the iterative scheme given by (3.7) converges to a solution. In the other words, we will find a relation between T and h, whose satisfaction guarantees convergence.

To proceed, consider the case when zero external heat is applied. Then (3.7) becomes in

$$u_{k+1,i} = \frac{T\kappa}{h^2} u_{k,i-1} + \left(1 - 2\,\frac{T\kappa}{h^2}\right) u_{k,i} + \frac{T\kappa}{h^2} u_{k,i+1}.$$

Also, replace $u_{k,i}$ by $g^k\, e^{j\,i\,\theta}$ to obtain

$$g^{k+1}\, e^{j\,i\,\theta} = \frac{T\kappa}{h^2} g^k \left(e^{j\,(i-1)\,\theta} + e^{j\,(i+1)\,\theta}\right) + \left(1 - 2\,\frac{T\kappa}{h^2}\right) g^k\, e^{j\,i\,\theta},$$

where θ is the spatial frequency and $j = \sqrt{-1}$. The parameter g is termed *the amplification factor* and the recurrence equation is stable if and only if $|g| \leq 1$, see [67] for details. Using Euler's formula and routine simplification now gives

$$g = \frac{T\kappa}{h^2} \left(e^{-j\,\theta} + e^{j\,\theta}\right) + \left(1 - 2\,\frac{T\kappa}{h^2}\right) = \frac{T\kappa}{h^2}\, 2\cos\theta + \left(1 - 2\,\frac{T\kappa}{h^2}\right).$$

Hence, $|g| \leq 1$ when

$$\frac{T}{h^2} < \frac{1}{2\,\kappa}. \tag{3.9}$$

3.5. NUMERICAL VERIFICATION OF A DISCRETE MODEL

For the verification process, a simulation is made. It is based on iterative solution to (3.7), which is placed into the body of a `for` cycle in Matlab. The continuous-time model (3.8) is discretised with a very short sampling period and then simulated like the discrete-time one.

3.5.1. Initial and boundary conditions

The initial condition of (3.7) and (3.8) is $u_{k,i}|_{t=0} = u_{0,i}$ and $u(t)_i|_{t=0} = u(0)_i$, respectively. The boundary conditions are $u(t,x)|_{x\in\partial\mathcal{D}}$, where $\partial\mathcal{D}$ denotes the boundary of the region where we find a solution. Since we consider a rod with n nodes, the boundary conditions of the discrete-time model are defined by $u_{k,1}, u_{k,n}$, and of the continuous-time model by $u(t)_1, u(t)_n$, $\forall\, k, t \geq 0$.

Let the initial condition be given in Fig. 3.5 and the boundary conditions be $u_{k,1} = u_{k,n} = 20\,°\mathrm{C}$ and $u(t)_1 = u(t)_n = 20\,°\mathrm{C}$.

3.5.2. Simulation results

Let $n = 59$. Put

$$\varkappa = 230\,\mathrm{W\,m^{-1}\,K^{-1}}, \qquad \rho = 2700\,\mathrm{kg\,m^{-3}}, \qquad c_p = 900\,\mathrm{J\,K^{-1}\,kg^{-1}} \tag{3.10}$$

and substitute

$$T = 1\,\mathrm{s}, \qquad h = \frac{1}{n} = \frac{1}{59}\,\mathrm{m}, \tag{3.11}$$

which satisfies (3.9). Now, (3.7) has the form

$$u_{k+1,i} = 0.33\,u_{k,i-1} + 0.34\,u_{k,i} + 0.33\,u_{k,i+1} + q_{k,i} \tag{3.12}$$

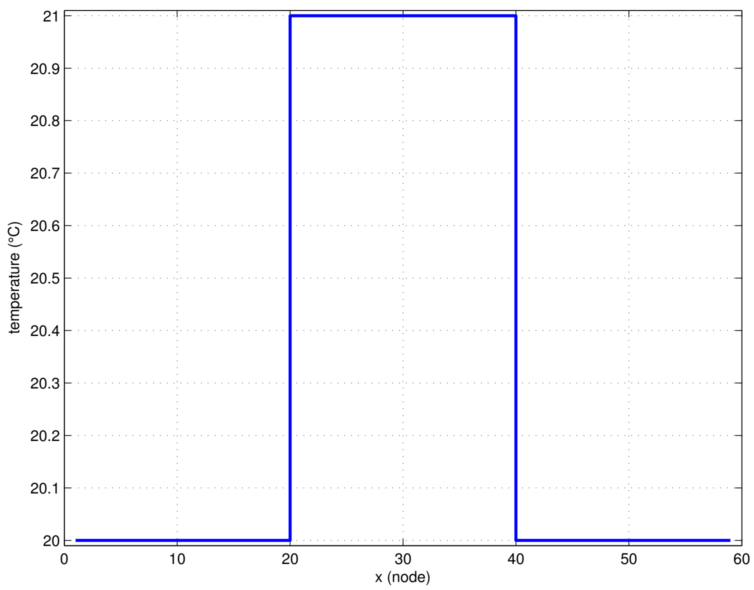

Fig. 3.5. Initial condition

and (3.8) has the form

$$(3.13)\quad \left(\frac{\partial u(t)}{\partial t}\right)_i = 0.33\,(u(t)_{i-1} - 2\,u(t)_i + u(t)_{i+1}) + f(t)_i.$$

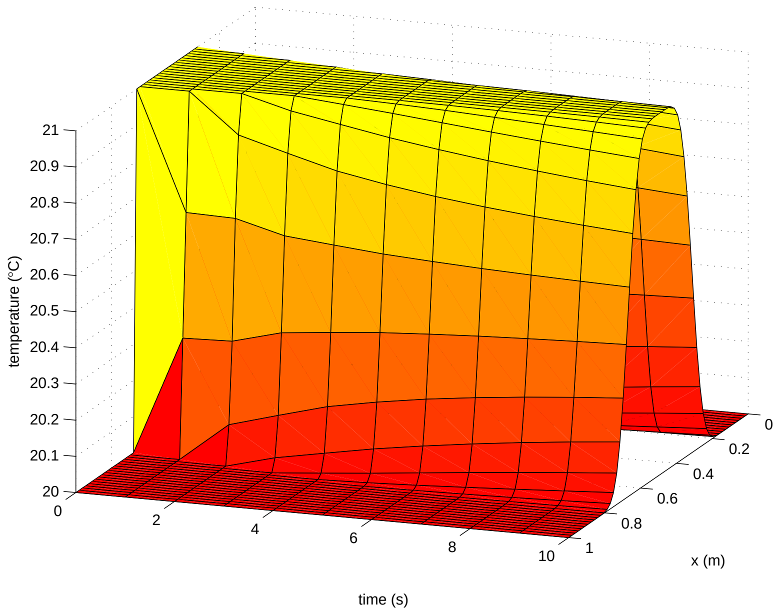

Fig. 3.6. Behaviour of the signal u when the condition (3.9) is satisfied

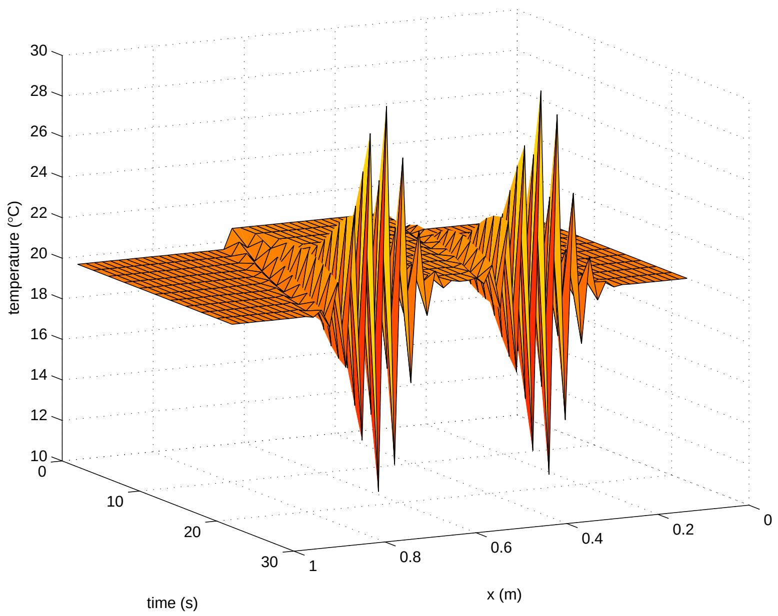

Fig. 3.7. Behaviour of the signal u when the condition (3.9) is not satisfied

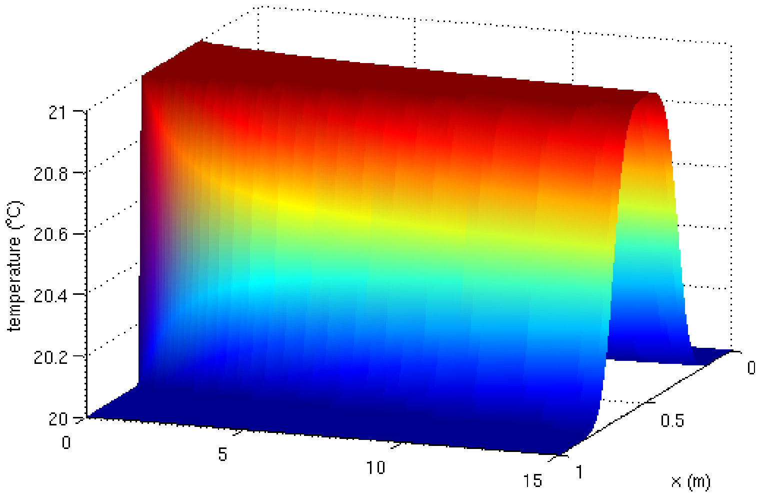

Fig. 3.8. Behaviour of the signal u in the continuous-time case

Results of simulation of (3.12) are shown in Fig. 3.6. Fig. 3.7 shows result of simulation when the condition (3.9) is not satisfied. One can see that the solution does not converge in this case. Results of simulation of (3.13) are shown in Fig. 3.8.

CHAPTER 4

Multivariate transfer functions for spatially invariant systems

In this chapter, we will derive a model useful for control design via polynomial technique – *transfer function*. We shall concentrate on the linear spatially distributed spatially and time invariant systems described by constant coefficient PDEs with temporal variable t and one spatial variable x. Extension to PDEs with more than one spatial variables is straightforward.

We assume infinite spatial domain. Of course, this assumption is never valid for a physical system, but it simplifies the design a lot and allows neglecting the boundary conditions.

4.1. CONTINUOUS-TIME CASE

In the single-input single-output continuous-time case, a linear spatially-distributed time-invariant system cab be represented by the input/output relationship

$$y(t,r) = \int_{-\infty}^{t} \left[\sum_{j=-\infty}^{\infty} g(t-\tau, r-j)\, u(\tau, j) \right] \mathrm{d}\tau \tag{4.1}$$

where $u(t,r) \in \mathbb{R}$ is the input at time $t \in \mathbb{R}$ and spatial point $r \in \mathbb{Z}$, $y(t,r) \in \mathbb{R}$ is the output at time $t \in \mathbb{R}$ and spatial point $r \in \mathbb{Z}$, and $g(t,r) \in \mathbb{R}$ is the value of the impulse response at time $t \in \mathbb{R}$ and spatial point $r \in \mathbb{Z}$. The upper limit in the integral is t since we assume causality in time, i. e., $g(t,r) = 0$ for all $t < 0$. We do not assume causality in the space, i. e., $g(t,r) \neq 0$ in general for all $r \in \mathbb{Z}$.

There is usually some damping along the distributed structure in the application to discretised PDE. Hence, we can assume that

$$\sum_{j=-\infty}^{\infty} |g(t,j)| < \infty \quad \text{for all} \quad t > 0.$$

If this assumption is satisfied and if the input u is a bounded function of the spatial variable r for every $t \in \mathbb{R}$, from (4.1) follows that output response y resulting from u is also bounded function of r for every $t \in \mathbb{R}$. Furthermore, we have

$$|y(t,r)| \leq \int_{-\infty}^{t} \left[\sum_{j=-\infty}^{\infty} |g(t-\tau, r-j)| \right] \left[\sup_{j\in\mathbb{Z}} |u(\tau, j)| \right] \mathrm{d}\tau$$

We see that if $\sum_j |g(t,j)|$ and $\sup_r |u(t,r)|$ are locally integrable functions of t, then (4.1) is well defined.

A use of the joint Laplace and z-transform leads to the transfer function of a system (4.1). Let $f(t,l)$ be a real-valued function defined on the Cartesian product $\mathbb{R} \times \mathbb{Z}$ and $f(t,l) = 0$ for $t < 0$. The joint Laplace and z-transform of $f(t,l)$ is given as

$$F(s,w) = \int_{t=0}^{\infty} \left[\sum_{l=-\infty}^{+\infty} f(t,l)\, w^{-l} \right] e^{-s\,t}\, \mathrm{d}t. \tag{4.2}$$

We assume that f is constrained, so that $F(s,w)$ is well defined for (s,w) belonging to some subset of $\mathbb{C} \times \mathbb{C}$. See also [10].

Consider a PDE as described in Sec. 3.1. Let u be an output and f be an input. Use one of the common discretisation schemes for spatial variable. Perform the transform (4.2) and manipulate to obtain the transfer function

$$P(s,w) = \frac{b(s,w)}{a(s,w)}. \tag{4.3}$$

In (4.3) the variable s corresponds to time and the variable w corresponds to shift along the spatial coordinate axis. Like in the discrete-time case, the polynomials a and b are *one-sided* in z and *two-sided* in w and can be assumed in the form

$$a(s,w) = \sum_{k=0}^{n} \sum_{l=0}^{m} a_{k,l} s^k \left(w^l + w^{-l} \right), \qquad n \geq 0, \quad m \geq 0 \tag{4.4}$$

and similarly for the polynomial b. Following the systems-over-rings concept, the notation

$$a[w](s) = a_n(w)s^n + a_{n-1}(w)s^{n-1} + \cdots + a_0(w) \tag{4.5}$$

can be used to emphasise that the polynomial a can be viewed as a polynomial in s with coefficients being functions of w.

4.1. Example. Consider (3.8). Let u be output and f input. Perform the transform (4.2) and manipulate to obtain the transfer function

$$P(s,w) = \frac{1}{s - \frac{\kappa}{h^2}\left(w - 2 + w^{-1}\right)}. \tag{4.6}$$

Substituting values of (3.10) and (3.11) gives

$$P(s,w) = \frac{1}{s - 0.33\,(w - 2 + w^{-1})}. \tag{4.7}$$

□

4.2. DISCRETE-TIME CASE

In the discrete-time single-input single-output case, a linear spatially-distributed time-invariant system cab be represented by the input/output convolution sum

$$y(l,r) = \sum_{i=-\infty}^{k} \sum_{j=-\infty}^{\infty} g(k-i, r-j)\, u(i,j),$$

where the input $u(k,r)$, output $y(k,r)$ and unit-pulse response $g(k,r)$ are all real-valued functions defined on the Cartesian product $\mathbb{Z} \times \mathbb{Z}$, k is discrete time variable and r is the discrete spatial variable. We assume causality in time, so, the sum on i terminates at $i = k$.

A sequence of two z-transforms can be performed to obtain the transfer function – one unilateral corresponding to t and the other bilateral corresponding to x. Let $f(k,l)$ be a real-valued function defined on the Cartesian product $\mathbb{Z}\times\mathbb{Z}$ and $f(k,l) = 0$ for all $k < 0$. The z-transform of $f(k,l)$ is given as

$$F(z,w) = \sum_{k=0}^{+\infty} \sum_{l=-\infty}^{+\infty} f(k,l)\, w^{-l}\, z^{k}. \tag{4.8}$$

We assume that f is constrained, so that $F(z,w)$ is well defined for (z,w) belonging to some subset of $\mathbb{C} \times \mathbb{C}$. See also [10].

Consider a PDE as described in Sec. 3.1. Let u be an output and f be an input. Use one of the common discretisation schemes (as it was shown in the previous chapter) to get a PRE. Perform the z-transform (4.8) and manipulate to obtain the transfer function

$$P(z,w) = \frac{b(z,w)}{a(z,w)}. \tag{4.9}$$

In (4.9) the variable z corresponds to time delay and the variable w corresponds to shift along the spatial coordinate axis. Since the system is causal in time and non-causal in space, the polynomials a and b are *one-sided* in z and *two-sided* in w. For physical systems, it is reasonable to assume spatial symmetry: the polynomial a can be assumed in the form

$$a(z,w) = \sum_{k=0}^{n} \sum_{l=0}^{m} a_{k,l} z^k \Big(w^l + w^{-l} \Big), \qquad n \geq 0, \quad m \geq 0 \tag{4.10}$$

and similarly for the polynomial b. Following the systems-over-rings concept, the notation

$$a[w](z) = a_n(w) z^n + a_{n-1}(w) z^{n-1} + \cdots + a_0(w) \tag{4.11}$$

can be used to emphasise that the polynomial a can be viewed as a polynomial in z with coefficients being functions of w.

4.2. Example. We obtain the transfer function corresponding to (3.7) with input q and output u. Simply, the z-transform and algebraic manipulation give

$$P(z,w) = \frac{z}{1 + \left(2\,\frac{T\kappa}{h^2} - 1\right) z - \frac{T\kappa}{h^2} \left(w + w^{-1}\right) z}. \tag{4.12}$$

Substituting values of (3.10) and (3.11) gives

$$P(z,w) = \frac{z}{1 - 0.34\, z - 0.33 \left(w + w^{-1}\right) z}. \tag{4.13}$$

□

Note the transfer function of thin flexible plate of Example 3.2 was obtained analogously in [2].

CHAPTER 5

Multidimensional BIBO stability analysis

Consider a system described by the transfer function (4.9) or (4.3). Stability of such a system can be studied by analysing roots of its denominator polynomial, with the first stability criterion given by Justice and Shanks [34]. This is similar to the lumped (1-D) case, but having two variables, the values of the denominator polynomial a must be studied on a combination of the unit circle and disc or a combination of the unit circle and a half-plane.

It was shown in [20] and discussed in e. g. [33, 15] that Shanks theorem loses its validity in the case when the system has a nonessential singularity of the second kind. The transfer function is said to have a nonessential singularity of the second kind if the relatively prime numerator and denominator share a common zero on the stability region boundary. In such a case the numerator affects the stability and the necessity condition of Shanks theorem does not hold. The following example was pointed out in [20], see also [33].

5.1. Example. Consider

$$G_1 = \frac{(1-z_1)^8(1-z_2)^8}{2-z_1-z_2}, \qquad G_2 = \frac{(1-z_1)(1-z_2)}{2-z_1-z_2}.$$

Both the above transfer functions have nonzero denominators on $\{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}$ except the point $z_1 = 1$, $z_2 = 1$ and have a nonessential singularity of the second kind at the point $z_1 = 1$, $z_2 = 1$. Goodman [20] proved that G_1 is BIBO stable and G_2 is BIBO unstable. □

A more complicated test should be used to check the stability precisely in the case when a system has a nonessential singularity of the second kind. One of such tests can be found in [71]. However, disregarding these uncommon situations by desiring a closed-loop system with no singularity on the stability region boundary, examining denominator polynomials suffices. In the following sections, we will deal with discrete-time and continuous-time case separately.

5.1. DISCRETE-TIME CASE

Specialising the general Shanks theorem to systems with spatio-temporal transfer functions, the classical results on stability follows.

5.2. Theorem. [10, Theorem 4.3, p. 126] A system described by (4.9) with the polynomials free of a common factor is BIBO stable if $a(z, w) \neq 0$ for all $\{|w| = 1\} \cap \{|z| \leq 1\}$. $\square$

An immediate reformulation of this test goes in the spirit of the concept of systems over rings.

5.3. Corollary. A system described by (4.9) with the polynomials free of a common factor is BIBO stable if $a[w](z) = a_n(w)z^n + a_{n-1}(w)z^{n-1} + \cdots + a_0(w)$ is stable (has its roots outside the open unit circle) for all $|w| = 1$. $\square$

A vast number of extensions and simplifications have been proposed in the last decades, such as [68, 46, 47, 29, 60], just to name a few. The key trick used in this chapter is described in [62]. It consists in establishing an equivalence of stability of a 2-D polynomial and positiveness of a certain symmetric polynomial matrix (Schur-Cohn matrix for discrete-time systems) on the unit circle. Apart from algebraic criteria like [61], the advanced toolset of linear matrix inequalities (LMIs) can be used to test positiveness of a polynomial matrix on a unit circle, see [72, 18, 19, 26, 16].

The Schur-Cohn matrix H for a polynomial $a(z, w)$ has the form

$$H(w) = S_1\, S_1^{\mathrm{T}} - S_2\, S_2^{\mathrm{T}}, \tag{5.1}$$

where

$$S_1 = \begin{pmatrix} a_0(w) & 0 & \cdots & 0 \\ a_1(w) & a_0(w) & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{n-1}(w) & a_{n-2}(w) & \cdots & a_0(w) \end{pmatrix},$$

$$S_2 = \begin{pmatrix} a_n(w) & 0 & \cdots & 0 \\ a_{n-1}(w) & a_n(w) & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_1(w) & a_2(w) & \cdots & a_n(w) \end{pmatrix}.$$

Examples of the Schur-Cohn matrix for $n = 1, 2, 3$ follow

$$H_{n=1}(w) = \left(\; {a_0}^2 - {a_1}^2 \; \right),$$

$$H_{n=2}(w) = \begin{pmatrix} {a_0}^2 - {a_2}^2 & a_0\, a_1 - a_1\, a_2 \\ a_0\, a_1 - a_1\, a_2 & {a_0}^2 - {a_2}^2 \end{pmatrix},$$

$$H_{n=3}(w) = \begin{pmatrix} {a_0}^2 - {a_3}^2 & a_0\, a_1 - a_2\, a_3 & a_0\, a_2 - a_1\, a_3 \\ a_0\, a_1 - a_2\, a_3 & {a_0}^2 + {a_1}^2 - {a_2}^2 - {a_3}^2 & a_0\, a_1 - a_2\, a_3 \\ a_0\, a_2 - a_1\, a_3 & a_0\, a_1 - a_2\, a_3 & {a_0}^2 - {a_3}^2 \end{pmatrix}.$$

See e. g. [9] for comprehensive overview. The following lemma formally states the key tool for this chapter.

5.4. Lemma. [62] A polynomial $a[w](z)$ of the form (4.10) is stable if and only if its Schur-Cohn matrix $H(w)$ is positive definite on the unit circle, that is, $H(w) \succ 0$ for all $|w| = 1$. $\square$

The Schur-Cohn matrix is a symmetric polynomial matrix $H(w) = H_0 + H_1(w + w^{-1}) + \cdots + H_{2\,m}(w^{2\,m} + w^{-2\,m})$ of size n. Using semidefinite programming formulation and the result stated in [72, 18], the matrix is positive definite for all $|w| = 1$ if and only if there exists a symmetric matrix M of size $2\,n\,m$ such that

$$\begin{aligned} L(M) = & \left(\begin{array}{c|ccc} H_0 & H_1 & \cdots & H_{2m} \\ \hline H_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{2m} & 0 & \cdots & 0 \end{array}\right) + \\ & + \left(\begin{array}{ccc} I & & \\ \hline & \ddots & \\ & & I \\ 0 & \cdots & 0 \end{array}\right) M \left(\begin{array}{c|ccc} I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{array}\right) - \\ & - \left(\begin{array}{ccc} 0 & \cdots & 0 \\ \hline I & & \\ & \ddots & \\ & & I \end{array}\right) M \left(\begin{array}{c|ccc} 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{array}\right) \succ 0. \end{aligned} \tag{5.2}$$

5.5. Example. Consider (4.13) with the denominator polynomial

$$a(z, w) = 1 - 0.34\, z - 0.33\, (w + w^{-1})\, z.$$

Following the above approach, we have

$$S_1 = 1, \qquad S_2 = -0.34 - 0.33\, (w + w^{-1})$$

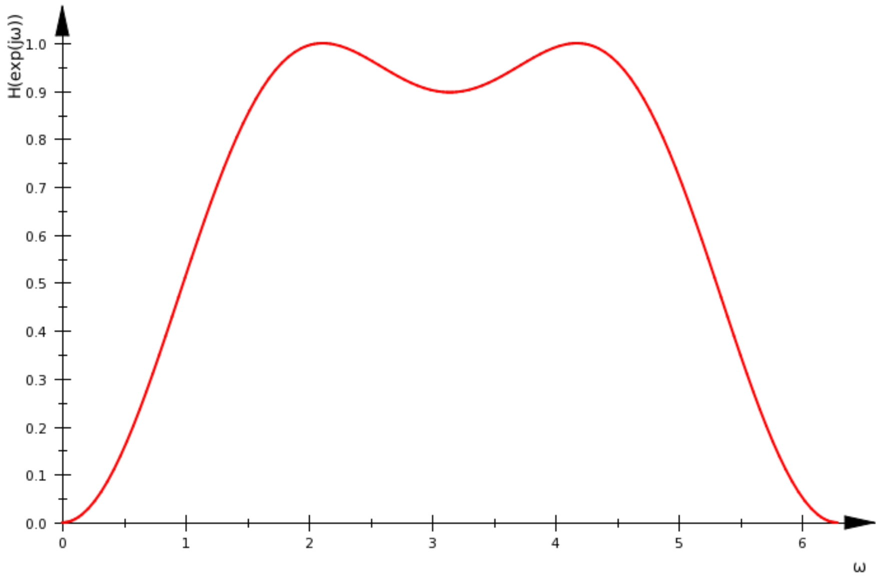

Fig. 5.1. Values of $H(w)$ for all $|w| = 1$

and the Schur-Cohn matrix

$$H(w) = 0.6666 - 0.2244\,(w + w^{-1}) - 0.1089\,(w^2 + w^{-2}).$$

Since $H(w)$ is scalar, we can easily check its positiveness graphically. Simply, plot values of $H(w)$ for all $|w| = 1$, i. e., plot values of $H(e^{j\,\omega})$ for all $\omega \in \langle 0; 2\,\pi \rangle$. The result is in Fig. 5.1. Since $H(1) = 0$, the system is not stable. $\square$

5.6. Example. Consider the control scheme of Fig. 5.2. Does the controller

$$R(z, w) = 0.1 + 0.2\,z + 0.3\,(w + w^{-1})\,z$$

stabilise the plant (4.13)?

The characteristic polynomial of the closed-loop system is

$$c(z, w) = 1 - 0.24\,z - 0.33\,(w + w^{-1})\,z + 0.2\,z^2 + 0.3\,(w + w^{-1})\,z^2.$$

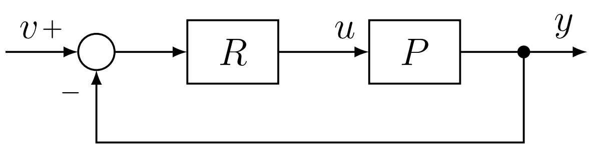

Fig. 5.2. Closed-loop system

Using the above approach, we get

$$S_1 = \begin{pmatrix} 1.0 & 0 \\ -0.24 - 0.33\,(w + w^{-1}) & 1.0 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 0.2 + 0.3\,(w + w^{-1}) & 0 \\ -0.24 - 0.33\,(w + w^{-1}) & 0.2 + 0.3\,(w + w^{-1}) \end{pmatrix},$$

$$H_0 = \begin{pmatrix} 0.78 & 0.006 \\ 0.006 & 0.78 \end{pmatrix}, \quad H_1 = \begin{pmatrix} -0.12 & -0.192 \\ -0.192 & -0.12 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{pmatrix}$$

Writing the following SeDuMi/Yalmip code

```
M1 = zeros(6,4);
M1(1,1) = 1.0;
M1(2,2) = 1.0;
M1(3,3) = 1.0;
M1(4,4) = 1.0;

M2 = zeros(6,4);
M2(3,1) = 1.0;
M2(4,2) = 1.0;
M2(5,3) = 1.0;
M2(6,4) = 1.0;

t0 = 4.0;
M=sdpvar(t0)

H=[0.78, 0.006, -0.12, -0.192, -0.1, 0.1; ...
    0.006, 0.78, -0.192, -0.12, 0.1, -0.1; ...
   -0.12, -0.192, 0, 0, 0, 0; -0.192, -0.12, 0, 0, 0, 0; ...
   -0.1, 0.1, 0, 0, 0, 0; 0.1, -0.1, 0, 0, 0, 0];

L=H+M1*M*M1'-M2*M*M2'
F=set(L>=0)
solvesdp(F)
```

we get the output

```
  yalmiptime: 0.2351
  solvertime: 0.0973
        info: 'No problems detected (SeDuMi-1.1)'
     problem: 0
      dimacs: [1.1674e-10 0 0 6.9305e-12 6.6147e-11 1.1577e-10]
```

Since some matrix M exists that (5.2) holds, the closed-loop system is stable. $\square$

5.2. CONTINUOUS-TIME CASE

In the continuous-time case, the situation is similar.

5.7. Theorem. [10, Theorem 4.3, p. 126] Spatially distributed system described by the transfer function (4.3) with the polynomials free of a common factor is BIBO stable if $a(s,w) \neq 0 \quad$ for all $\quad \{|w| = 1\} \cap \{\Re\{s\} \geq 0\}$. □

An immediate reformulation of this test goes in the spirit of the concept of systems over rings.

5.8. Corollary. A described by (4.3) with the polynomials free of a common factor is BIBO stable if $a[w](s) = a_n(w)\, s^n + a_{n-1}(w)\, s^{n-1} + \cdots + a_0(w)$ is stable (has its roots in the left half-plane) for all $|w| = 1$. □

In the continuous-time case, Hermite-Fujiwara matrix plays a role in the stability testing. Let $a^*(s)$ denote $a(-s)$. The Hermite-Fujiwara matrix is defined as $H = (h_{ij})_{i,j=1,\ldots,n}$, where

$$h_{ij} = (-1)^{j-1} \sum_{k=1}^{m_{ij}} a_{j+k-1}\, a^*_{i-k} - a_{i-k}\, a^*_{j+k-1},$$

where $m_{ij} = \min(i, n-j+1)$. Examples of the Hermite-Fujiwara matrix corresponding to $a[w](s)$ for $n = 1, 2, 3, 4$ follow

$$H_{n=1}(w) = \begin{pmatrix} 2\, a_0\, a_1 \end{pmatrix}, \qquad H_{n=2}(w) = \begin{pmatrix} 2\, a_0\, a_1 & 0 \\ 0 & 2\, a_1\, a_2 \end{pmatrix},$$

$$H_{n=3}(w) = \begin{pmatrix} 2\, a_0\, a_1 & 0 & 2\, a_0\, a_3 \\ 0 & 2\, a_1\, a_2 - 2\, a_0\, a_3 & 0 \\ 2\, a_0\, a_3 & 0 & 2\, a_2\, a_3 \end{pmatrix},$$

$$H_{n=4}(w) = \begin{pmatrix} 2\, a_0\, a_1 & 0 & 2\, a_0\, a_3 & 0 \\ 0 & 2\, a_1\, a_2 - 2\, a_0\, a_3 & 0 & 2\, a_1\, a_4 \\ 2\, a_0\, a_3 & 0 & 2\, a_2\, a_3 - 2\, a_1\, a_4 & 0 \\ 0 & 2\, a_1\, a_4 & 0 & 2\, a_3\, a_4 \end{pmatrix}.$$

The bellow lemma is reformulation of Lemma 5.4 for the continuous-time systems.

5.9. Lemma. A polynomial $a[w](s)$ of the form (4.4) is stable if and only if its Hermite-Fujiwara matrix $H(w)$ is positive definite on the unit circle, that is, $H(w) \succ 0$ for all $|w| = 1$. □

The Hermite-Fujiwara matrix is a symmetric polynomial matrix $H(w) = H_0 + H_1(w + w^{-1}) + \cdots + H_{2m}(w^{2m} + w^{-2m})$ of size n. The same result [72, 18] that was used for testing positiveness of the Schur-Cohn matrix on the unit circle can be used here.

5.3. CONCLUDING REMARKS

The LMI formalism offers easy extension from analysis to constructive synthesis. However, the obvious obstacle that prevents us from using the LMIs directly is the bilinear dependence of the coefficients of the Schur-Cohn and the Hermite-Fujiwara matrices on the coefficients of the original polynomials, see e.g. [28]. Sec. 6.1 and 6.2 offer a partial solution.

CHAPTER 6

Polynomial approach to control

This chapter proposes some control design techniques for spatially distributed systems based on operations with polynomials. In the first two sections, the method presented in Chap. 5 is further developed and a technique for stabilisation of a spatially distributed system is proposed. In the next sections, LQG control, $\mathcal{H}_2$-optimal control and dead-beat control of spatially distributed systems are developed.

In what follows, stabilisable systems free of hidden modes are considered. A system described by the transfer function (4.9) is stabilisable if and only if the polynomials a and b have no unstable zero in common [59].

6.1. POSITIVE POLYNOMIAL APPROACH TO STABILISATION: DISCRETE-TIME CASE

This section deals with stabilisation of a discrete-time spatially distributed system. We consider a system described by the transfer function (4.9). Consider the control scheme of Fig. 6.1, $a, b, c, x, y \in \mathbb{R}[w][z]$ and a controller given by transfer function

$$R(z,w) = \frac{y(z,w)}{x(z,w)}. \tag{6.1}$$

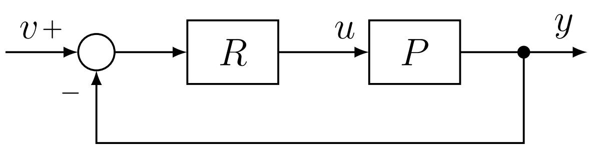

Fig. 6.1. Control scheme

The closed-loop system is stable if and only if the characteristic polynomial

$$a\,x + b\,y = c \tag{6.2}$$

is stable. In other words, solving (6.2) for polynomials x and y with given a and b and a stable polynomial c makes the closed-loop system stable. Suppose spatial symmetry of the closed-loop system. Let c have the form

$$c = \sum_{k=0}^{\hat{n}} \sum_{l=0}^{\hat{m}} c_{k,l}\, z^k \left(w^l + w^{-l}\right), \qquad \hat{n} \geq 0, \quad \hat{m} \geq 0. \tag{6.3}$$

Now, the problem arises how to find coefficients $c_{k,l}$ of a polynomial c that it is stable. The LMI-based approach as described in Chap. 5 cannot be used directly, since (5.2) depends non-linearly on polynomial coefficients, i. e. on matrices S_1 and S_2, which play a role in stability.

First, we shall analyse the case when the system is of order one in the time variable. We will discuss the general case later on.

6.1.1. Systems of order one in the time variable

In this subsection, we assume the following.

6.1. Assumption. The system is of order one in the time variable. Hence, the polynomial (4.10) is of degree $n = 1$ in the variable z. □

This assumption is not so restrictive. Many processes and phenomena in the nature can be described by the parabolic PDE (3.1), which is of first order with respect to time.

Extending the well-known results on solvability of a Diophantine equation in the 1-D setting [41], it is clear that the closed-loop polynomial c has to be of degree $2\,n - 1$ or greater in the variable z to design a realisable controller. Since $n = 1$, $\hat{n} \geq 2\,n - 1 = 1$. Thus, let $\hat{n} = 1$. Our task simplifies to find all stable polynomials with degree equal to 1 in z. Degree in w can be arbitrary.

Due to the above assumptions, S_1 and S_2 are now scalars. However, for lucidity, the notation for matrices is kept in what follows. We propose the following theorem. Without a loss of generality we suppose $a_0(w) > 0$, i. e. $S_1 > 0$.

6.2. Theorem. A polynomial (6.3) with $\hat{n} = 1$ is stable if and only if

$$\begin{pmatrix} S_1 & S_2 \\ S_2^{\mathrm{T}} & S_1^{\mathrm{T}} \end{pmatrix} \succ 0, \tag{6.4}$$

where $H(w) = S_1 S_1^{\mathrm{T}} - S_2 S_2^{\mathrm{T}}$ is the Schur-Cohn matrix corresponding to c.

P r o o f. It follows from Sylvester's criterion that (6.4) holds if and only if

$$S_1 > 0 \qquad \text{and} \qquad \det \begin{pmatrix} S_1 & S_2 \\ S_2^{\mathrm{T}} & S_1^{\mathrm{T}} \end{pmatrix} > 0.$$

The former condition was assumed without a loss of generality before, the latter is equal to $S_1 S_1^{\mathrm{T}} - S_2 S_2^{\mathrm{T}} > 0$. Use of Lemma 5.4 concludes the proof. □

The left-hand side matrix in (6.4) is a symmetric trigonometric polynomial matrix $H(w) = H_0 + H_1(w + w^{-1}) + \cdots + H_{\hat{m}}(w^{\hat{m}} + w^{-\hat{m}})$ of size $2\,\hat{n}$. It is positive semidefinite for $|w| = 1$ if and only if there exists a symmetric matrix M of size $2\,\hat{n}\,\hat{m}$ such that

$$L(M, H_0, H_1, \ldots, H_{\hat{m}}) = \left(\begin{array}{c|ccc} H_0 & H_1 & \cdots & H_{\hat{m}} \\ \hline H_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{\hat{m}} & 0 & \cdots & 0 \end{array}\right) +$$

$$(6.5) \qquad + \left(\begin{array}{ccc} I & & \\ \hline & \ddots & \\ & & I \\ 0 & \cdots & 0 \end{array}\right) M \left(\begin{array}{c|ccc} I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{array}\right) -$$

$$- \left(\begin{array}{ccc} 0 & \cdots & 0 \\ \hline I & & \\ & \ddots & \\ & & I \end{array}\right) M \left(\begin{array}{c|ccc} 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{array}\right) \succeq 0,$$

where in contrast with (5.2) L depends also on H.

Theorem 6.2 allows us now to complete the following lemma.

6.3. Lemma. Consider a plant described by (3.1) with the transfer function (4.9). A controller with the transfer function (6.1) stabilises the plant if

$$a\,x + b\,y = c$$

is a such polynomial that (6.5) holds with

$$H(w) = \begin{pmatrix} S_1 & S_2 \\ S_2^{\mathrm{T}} & S_1^{\mathrm{T}} \end{pmatrix},$$

where $S_1 S_1^{\mathrm{T}} - S_2 S_2^{\mathrm{T}}$ is the Schur-Cohn matrix corresponding to c.

P r o o f. Follows immediately from Theorem 6.2 and the fact that c is the characteristic polynomial of closed-loop system. $\square$

By means of an example we apply the presented approach to a system of heat conduction in a rod.

6.4. Example. Consider the control scheme of Fig. 6.1, a system described by the transfer function (4.13) and a controller with the transfer function

$$R(z,w) = r_0 + r_1(w + w^{-1}), \tag{6.6}$$

where r_0 and r_1 are real constants. The characteristic polynomial of closed-loop system has the form

$$c(z,w) = 1 + (r_0 - 0.34)z + (r_1 - 0.32)\,(w + w^{-1})z. \tag{6.7}$$

Next, we find the constants r_0 and r_1 so that (6.7) is stable. The method described in Sec. 6.1.1 gives

$$\begin{aligned}
S_1 &= 1,\\
S_2 &= r_0 + \left(r_1 - \tfrac{8}{25}\right)(w + w^{-1}) - \tfrac{17}{50},\\
H_0 &= \begin{pmatrix} 1 & r_0 - \frac{17}{50} \\ r_0 - \frac{17}{50} & 1 \end{pmatrix},\\
H_1 &= \begin{pmatrix} 0 & r_1 - \frac{8}{25} \\ r_1 - \frac{8}{25} & 0 \end{pmatrix}.
\end{aligned}$$

Using SeDuMi and Yalmip [69, 44] we can check that some matrix M in (6.5) exists and r_0, r_1 are for example

$$r_0 = 0.2, \qquad r_1 = 0.28.$$

The characteristic polynomial c is

$$c = 1 - 0.14\,z - 0.04(w + w^{-1})z. \tag{6.8}$$

Since we know the structure of the controller, we can only substitute values of r_0 and r_1 into (6.6) and get

$$R = 0.2 + 0.28(w + w^{-1}).$$

In this case, we do not have to solve (6.2).

For illustration, in Fig. 6.2 we sketched all values of r_0 and r_1 for which the polynomial (6.7) is stable. A few numerical simulations follow. Fig. 3.5 and Fig. 3.6 give the initial condition and the response of the uncontrolled system, respectively. The response of the controlled plant and manipulated variable is shown in Fig. 6.3 and 6.4, respectively. □

6.5. Remark. With SeDuMi/Yalmip we can easily find a controller optimal in the sense of minimising a criterion. The criterion should be linear in variables used in the controller design (in the above example r_0, r_1). Some function (e. g. trace), are allowed, see [69, 44] for details. If a criterion is specified, solver returns the optimal solution. This is useful only when we are able to give such a function of controller constants whose minimisation has a good effect on the control quality. This optimisation technique is used in Example 6.12. □

6.1.2. The general case

In the general case, the situation is more complicated. Theorem 6.2 no longer holds. In this subsection, we show a possible way of stabilisation of a spatially distributed system of order greater than one.

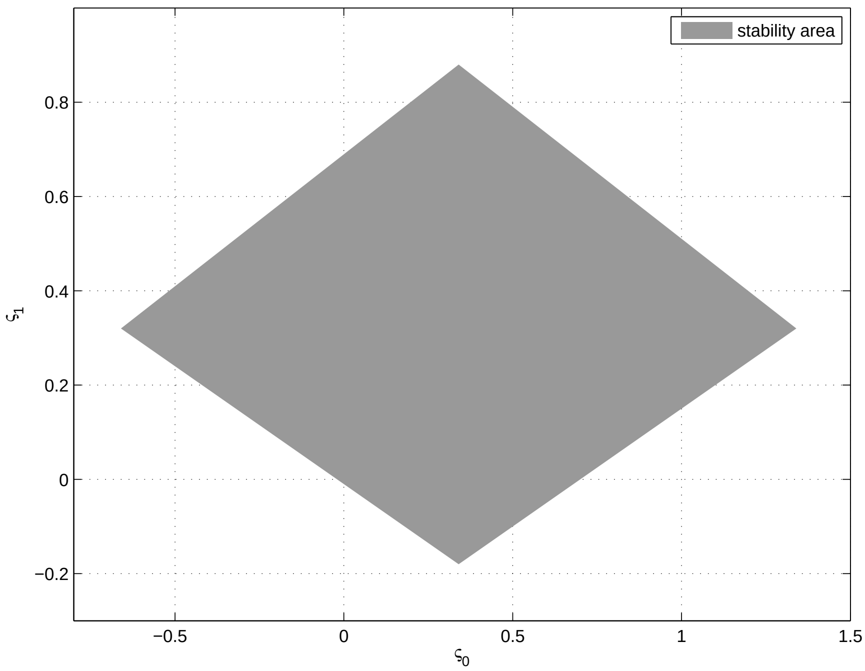

Fig. 6.2. The values of r_0 and r_1 for which the polynomial (6.7) is stable

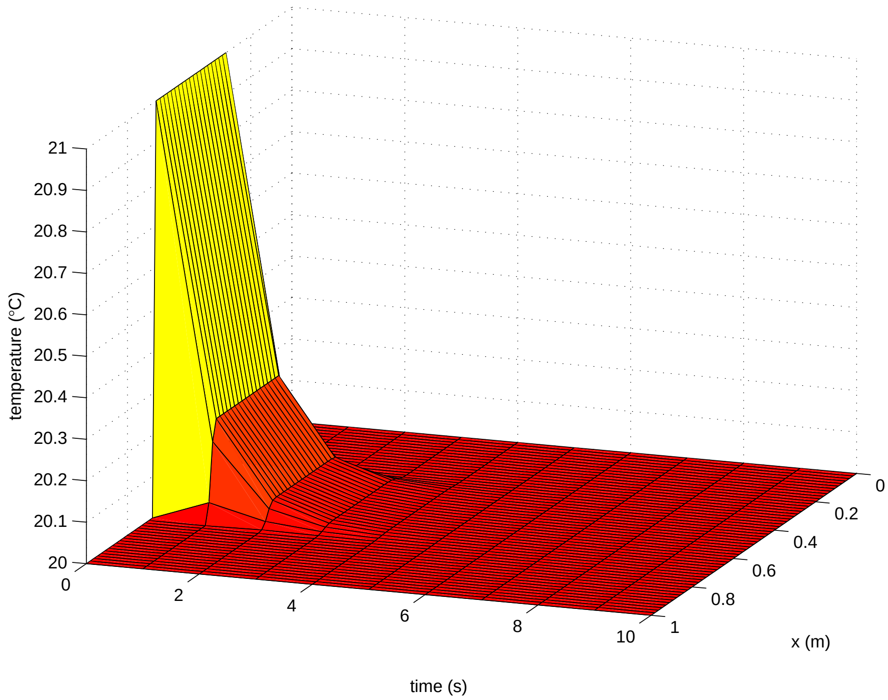

Fig. 6.3. Output of the plant, control applied

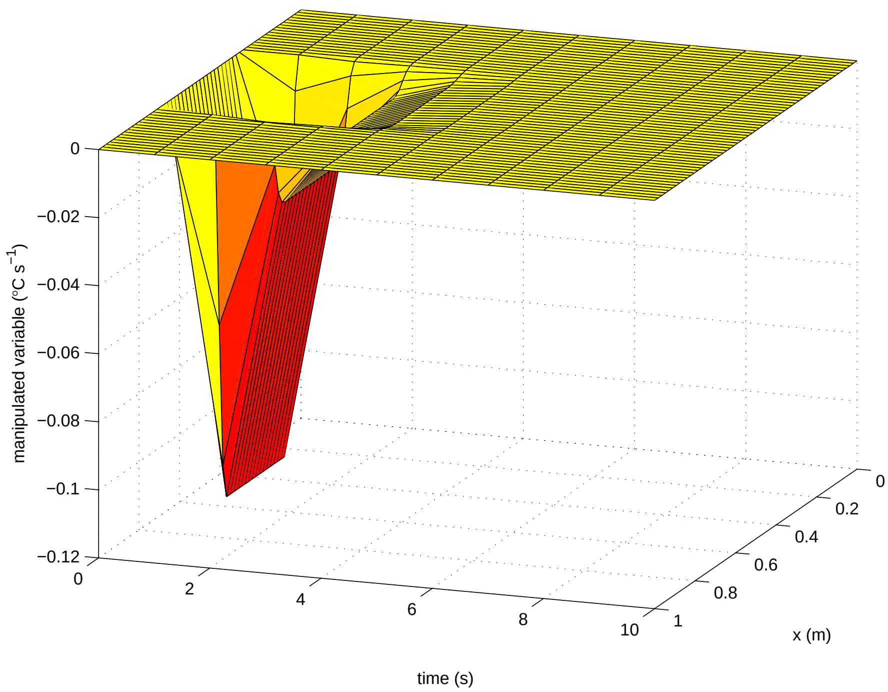

Fig. 6.4. Manipulated variable

We have the sufficient and necessary *non-convex* condition for stability of a polynomial c in the form

$$(6.9)\qquad S_1(c)\,S_1^{\mathrm{T}}(c) - S_2(c)\,S_2^{\mathrm{T}}(c) \succ 0,$$

where the argument means that the matrices correspond to the polynomial c. We are able to derive a *convex* condition, however, the new condition will be sufficient only. How to do this is shown in e. g. [30, 27, 53].

In [53], (6.9) is linearised. Using the Schur complement argument we obtain the condition

$$\begin{pmatrix} S_1(c)\,S_1^{\mathrm{T}}(c) & S_2(c) \\ S_2^{\mathrm{T}}(c) & I \end{pmatrix} \succ 0,$$

which is equivalent to (6.9). The non-convexity is now concerned in the upper-left block. Now, let S_3 be any matrix with corresponding dimension, then

$$(S_1(c) - S_3)\,(S_1(c) - S_3)^{\mathrm{T}} \succeq 0,$$

so that

$$S_1(c)\,S_1^{\mathrm{T}}(c) \succeq S_1(c)\,S_3^{\mathrm{T}} + S_3\,S_1^{\mathrm{T}}(c) + S_3\,S_3^{\mathrm{T}}.$$

The above right-hand side is affine transformation. If the LMI

$$\begin{pmatrix} S_1(c)\,S_3^{\mathrm{T}} + S_3\,S_1^{\mathrm{T}}(c) + S_3\,S_3^{\mathrm{T}} & S_2(c) \\ S_2^{\mathrm{T}}(c) & I \end{pmatrix} \succ 0$$

is satisfied, (6.9) is satisfied too. We proved the following lemma.

6.6. Lemma. A polynomial c of the form (6.3) is stable if

$$(6.10)\qquad \begin{pmatrix} S_1\,S_3^{\mathrm{T}} + S_3\,S_1^{\mathrm{T}} + S_3\,S_3^{\mathrm{T}} & S_2 \\ S_2^{\mathrm{T}} & I \end{pmatrix} \succ 0,$$

where $H(w) = S_1 S_1^{\mathrm{T}} - S_2 S_2^{\mathrm{T}}$ is the Schur-Cohn matrix corresponding to c and S_3 is a matrix of adequate dimension. □

The question comes out how to choose the matrix S_3. A possible way is as follows. Consider a *stable* polynomial

$$q = \sum_{k=0}^{\tilde{n}} \sum_{l=0}^{\tilde{m}} q_{k,l}\, z^k \Big(w^l + w^{-l}\Big), \qquad \tilde{n} \geq 0, \quad \tilde{m} \geq 0$$

and let $S_3 = S_1(q)$. Unfortunately, to find a stable polynomial is exactly what we try to do in this section. However, if a goal is the stabilisation only, we can put $q = 1$. Note that this technique does not allow use the optimisation that was used in the case of low-order systems, see Rem. 6.5.

6.7. Example. Consider a plant given by the transfer function

$$P(z,w) = \frac{z}{15 - 24\,z + 2(w + w^{-1})\,z + 15\,z^2 + (w^2 + w^{-2})\,z^2}$$

and a control with the fixed structure

$$R(z,w) = \frac{y_0 + y_1\,(w + w^{-1})}{x_0 + x_1\,(w + w^{-1})\,z}.$$

Find the constants x_0, x_1, y_0, y_1 such the closed-loop system is stable. Choose $S_3 = I$. The above method leads to (6.5) of size 24×24 with $H_0, \dots, H_3$ given in Fig. 6.5. Using SeDuMi/Yalmip one can check that some matrix M exists and the controller constants are for example

$$x_0 = 0.12, \quad x_1 = 0.002, \quad y_0 = 3.2, \quad y_1 = 0.32.$$

□

6.2. POSITIVE POLYNOMIAL APPROACH TO STABILISATION: CONTINUOUS-TIME CASE

This section deals with stabilisation of a continuous-time spatially distributed system. We consider a system described by the transfer function (4.3). Consider the control scheme of Fig. 6.1, $a, b, c, x, y \in \mathbb{R}[w][s]$ and a controller given by the transfer function

$$R(s,w) = \frac{y(s,w)}{x(s,w)}. \tag{6.11}$$

The characteristic polynomial determining the stability of the closed loop is $a\,x + b\,y = c$,

$$c = \sum_{k=0}^{\hat{n}} \sum_{l=0}^{\hat{m}} c_{k,l}\, s^k \Big(w^l + w^{-l}\Big), \qquad \hat{n} \geq 0, \quad \hat{m} \geq 0. \tag{6.12}$$

Like in the discrete-time case, the closed-loop polynomial c has to be of degree $2\,n - 1$ or greater in the variable s to design a realisable controller. Since $n = 1$,

$$
\begin{pmatrix} H_0 \\ H_1 \\ H_2 \\ H_3 \end{pmatrix} =
\left(\begin{array}{cccccc}
30\,x_0+1 & y_0-24\,x_0 & 15\,x_0-4\,x_1 & 0 & 0 & 0 \\
y_0-24\,x_0 & 30\,x_0+1 & y_0-24\,x_0 & 15\,x_0-4\,x_1 & 0 & 0 \\
15\,x_0-4\,x_1 & y_0-24\,x_0 & 30\,x_0+1 & y_0-24\,x_0 & 15\,x_0-4\,x_1 & 0 \\
0 & 15\,x_0-4\,x_1 & y_0-24\,x_0 & 1 & 0 & 0 \\
0 & 0 & 15\,x_0-4\,x_1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & y_1-2\,x_0+15\,x_1 & -23\,x_1 & 15\,x_1 & 0 & 0 \\
y_1-2\,x_0+15\,x_1 & 0 & y_1-2\,x_0+15\,x_1 & -23\,x_1 & 15\,x_1 & 0 \\
-23\,x_1 & y_1-2\,x_0+15\,x_1 & 0 & y_1-2\,x_0+15\,x_1 & -23\,x_1 & 15\,x_1 \\
15\,x_1 & -23\,x_1 & y_1-2\,x_0+15\,x_1 & 0 & 0 & 0 \\
0 & 15\,x_1 & -23\,x_1 & 0 & 0 & 0 \\
0 & 0 & 15\,x_1 & 0 & 0 & 0 \\
\hline
0 & x_0 & -2\,x_1 & 0 & 0 & 0 \\
x_0 & 0 & x_0 & -2\,x_1 & 0 & 0 \\
-2\,x_1 & x_0 & 0 & x_0 & -2\,x_1 & 0 \\
0 & -2\,x_1 & x_0 & 0 & 0 & 0 \\
0 & 0 & -2\,x_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & x_1 & 0 & 0 & 0 \\
0 & 0 & 0 & x_1 & 0 & 0 \\
x_1 & 0 & 0 & 0 & x_1 & 0 \\
0 & x_1 & 0 & 0 & 0 & 0 \\
0 & 0 & x_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

Fig. 6.5. The matrices $H_0, \ldots, H_3$

$\hat{n} \geq 2n - 1 = 1$. However, in the continuous-time case, we do not have to be so strict. Throw this subsection, we assume the following.

6.8. Assumption. The system is of order one in the time variable. The controller (6.11) is of at the most first order in the time. Hence, assume $\hat{n} = 2$ and the polynomial (6.12) in the form $c = c_2(w)\, s^2 + c_1(w)\, s + c_0(w)$. The degree of c in w can be arbitrary. □

The sufficient and necessary condition for stability of c can now be expressed by this lemma.

6.9. Lemma. A polynomial c of the form (6.12) with $\hat{n} = 2$ is stable if and only if

$$\begin{pmatrix} c_0(w) & 0 & 0 \\ 0 & c_1(w) & 0 \\ 0 & 0 & c_2(w) \end{pmatrix} \succ 0 \tag{6.13}$$

for all $|w| = 1$.

P r o o f. Follows immediately from Theorem 5.7 and the well-known sufficient and necessary condition for stability of polynomial of degree 2. □

The left-hand side of (6.13) is a symmetric trigonometric polynomial matrix $H(w) = H_0 + H_1(w + w^{-1}) + \cdots + H_{\hat{m}}(w^{\hat{m}} + w^{-\hat{m}})$ of size $2\,\hat{n}$. The method using (6.5) can be used to check its positiveness.

6.10. Remark. Generally, for $\hat{n} > 2$, an analogous condition

$$\operatorname{diag}\,[c_0(w), c_1(w), \ldots, c_{\hat{n}}(w)] \succ 0$$

is no longer sufficient. □

By means of an example we apply the above approach to a system of heat conduction in a rod.

6.11. Example. Consider the control scheme of Fig. 6.1, the plant (4.7), the controller (6.11), and the characteristic polynomial of the closed-loop system in the form

$$c(s, w) = c_2(w)\, s^2 + c_1(w)\, s + c_0(w), \tag{6.14}$$

where

$$c_0(w) = \sum_{l=0}^{\hat{m}} c_{0,l}\Big(w^l + w^{-l}\Big), \qquad c_1(w) = \sum_{l=0}^{\hat{m}} c_{1,l}\, s\Big(w^l + w^{-l}\Big),$$

$$c_2(w) = \sum_{l=0}^{\hat{m}} c_{2,l}\, s^2\Big(w^l + w^{-l}\Big).$$

Build (6.13) and solve (6.5). Using SeDuMi/Yalmip confirms that a matrix M exists and returns

$$c_0(w) = 2, \qquad c_1(w) = 2, \qquad c_2(w) = 2.$$

So, (6.14) has now the form

$$c(s,w) = 2\,s^2 + 2\,s + 2.$$

Now, solve $a\,x + b\,y = c$ for x and y. A solution is

$$\begin{aligned} x &= (0.5\,w + 3 + 0.5\,w^{-1})\,s + (0.5\,w + 3 + 0.5\,w^{-1}) \\ y &= (0.165\,w^2 + 0.66\,w - 1.65 + 0.66\,w^{-1} + 0.165\,w^{-2})\,s + \\ &\quad + (0.165\,w^2 + 0.66\,w - 1.65 + 0.66\,w^{-1} + 0.165\,w^{-2})\,. \end{aligned}$$

Responses to initial conditions of Fig. 3.5 of the closed-loop system are shown in Figs. 6.6 and 6.7. □

6.12. Example. Consider the control scheme of Fig. 6.1, the plant (4.7) and a controller with the transfer function

$$R(s,w) = \frac{y(s,w)}{x(s,w)} = \frac{y_{00} + y_{11}\,s\,(w + w^{-1})}{x_{00} + x_{10}\,s + x_{01}(w + w^{-1})}, \tag{6.15}$$

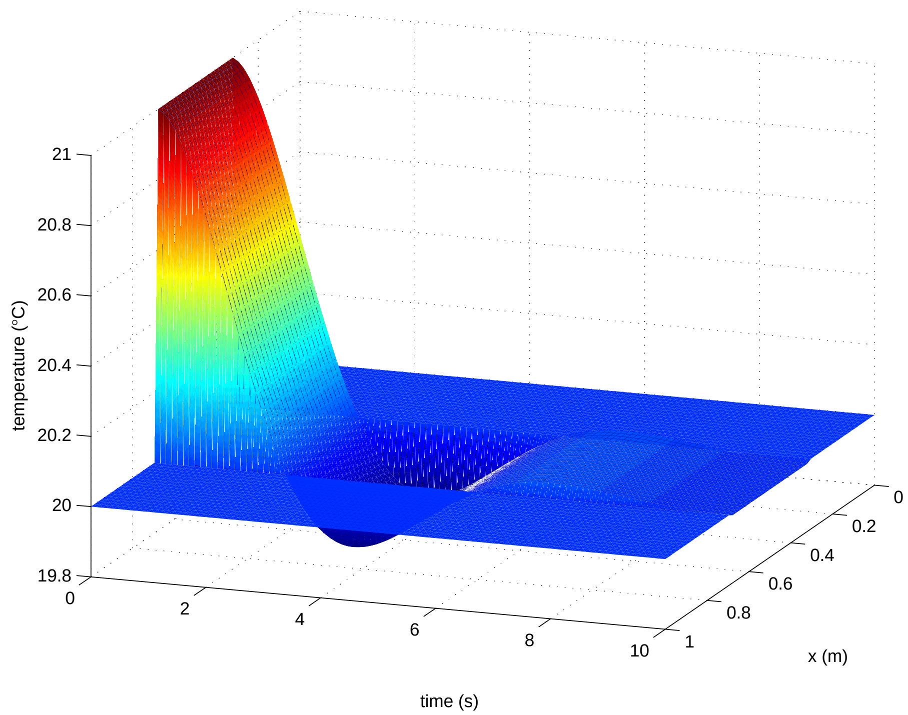

Fig. 6.6. Output of the plant, control applied

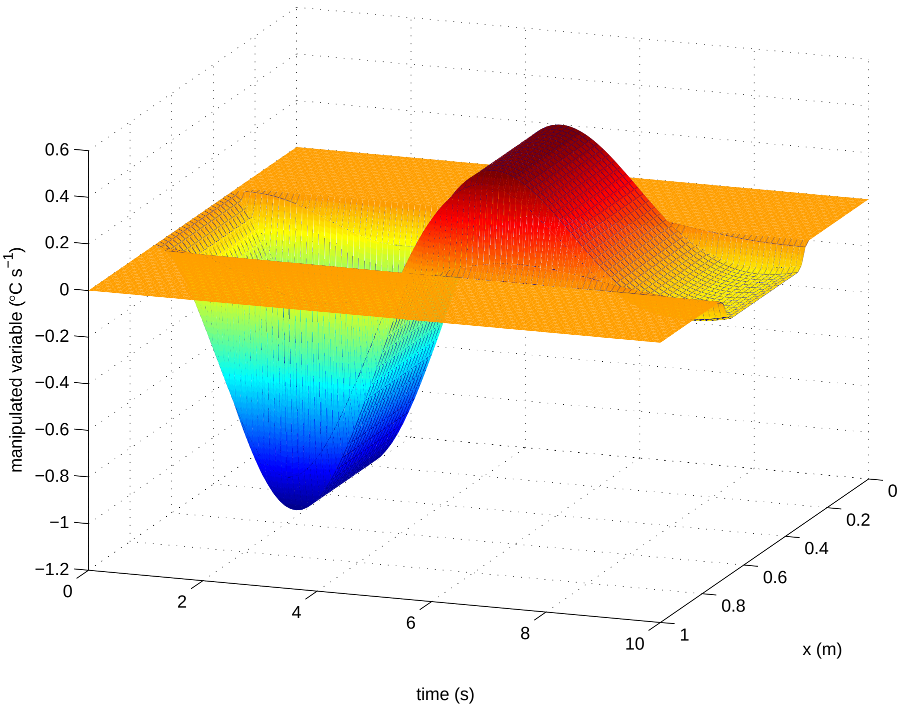

Fig. 6.7. Manipulated variable, control applied

where $y_{00}, y_{11}, x_{00}, x_{10}, x_{01}$ are real constants, which have to be found such that the closed-loop system is stable. The method described in this section gives

$$
\begin{aligned}
c_0(w) &= 0.66\,x_{00} - 0.66\,x_{01} + y_{00} - 0.33\,x_{00}(w + w^{-1}) + \\
&\quad +0.66\,x_{01}(w + w^{-1}) - 0.33\,x_{01}(w^2 + w^{-2}), \\
c_1(w) &= x_{00} + 0.66\,x_{10} + (-0.33\,x_{10} + x_{01} + y_{11})(w + w^{-1}), \\
c_2(w) &= x_{10}.
\end{aligned}
$$

Substituting the above polynomials into (6.13) and using SeDuMi [69] and Yalmip [44] we can check that some matrix M in (6.5) exists, coefficients of x and y are for example

$$
\begin{aligned}
&y_{00} = 2.6, \quad y_{11} = 0.7, \\
&x_{00} = 0.9, \quad x_{10} = 3, \quad x_{01} = 0.3
\end{aligned}
$$

and the corresponding controller is

$$
R_1(s, w) = \frac{2.6 + 0.7\,s\,(w + w^{-1})}{0.9 + 3\,s + 0.3\,(w + w^{-1})}. \tag{6.16}
$$

The Matlab/Yalmip code which leads to the above controller follows.

```
29 M1 = zeros(9,6);
30 M1(1,1) = 1.0;
31 M1(2,2) = 1.0;
32 M1(3,3) = 1.0;
33 M1(4,4) = 1.0;
34 M1(5,5) = 1.0;
35 M1(6,6) = 1.0;
36
37 M2 = zeros(9,6);
38 M2(4,1) = 1.0;
39 M2(5,2) = 1.0;
40 M2(6,3) = 1.0;
41 M2(7,4) = 1.0;
42 M2(8,5) = 1.0;
43 M2(9,6) = 1.0;
44
45 t0 = 6.0;
46 sdpvar x00 x10 x01 y00 y11
47 M=sdpvar(t0)
48
49 H=[0.66*x00 - 0.66*x01 + 1.0*y00, 0, 0, 0.66*x01 - 0.33*x00, ...
50   0, 0, -0.33*x01, 0, 0; ...
51   0, 1.0*x00 + 0.66*x10, 0, 0, 1.0*x01 - 0.33*x10 + 1.0*y11, ...
52   0, 0, 0, 0; ...
53   0, 0, 1.0*x10, 0, 0, 0, 0, 0, 0; 0.66*x11 - 0.33*x00, ...
54   0, 0, 0, 0, 0, 0, 0, 0; ...
55   0, 1.0*x01 - 0.33*x10 + 1.0*y11, 0, 0, 0, 0, 0, 0, 0; ...
56   0, 0, 0, 0, 0, 0, 0, 0, 0; -0.33*x01, 0, 0, 0, 0, 0, 0, 0, 0;...
57   0, 0, 0, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 0, 0, 0, 0, 0];
58
59 L=H+M1*M*M1'-M2*M*M2'
60 F=set(L>=0)
61 solvesdp(F)
```

Like before we will watch responses to initial condition of Fig. 3.5. Fig. 3.8 shows the response of the uncontrolled system. The response of controlled plant and manipulated variable is shown in Fig. 6.8 and 6.9, respectively. The closed--loop system is stable, however, responses could have a shorter settling time. Let us try to stabilise the system and maximise $|x_{10}|$ simultaneously. This corresponds to a change of line 61 to

```
62 solvesdp(F, -abs(x10))
```

Now, we get

$$
\begin{aligned}
&y_{00} = 18633, \quad y_{11} = -2231, \\
&x_{00} = 15494, \quad x_{10} = 10000, \quad x_{01} = 5531
\end{aligned}
$$

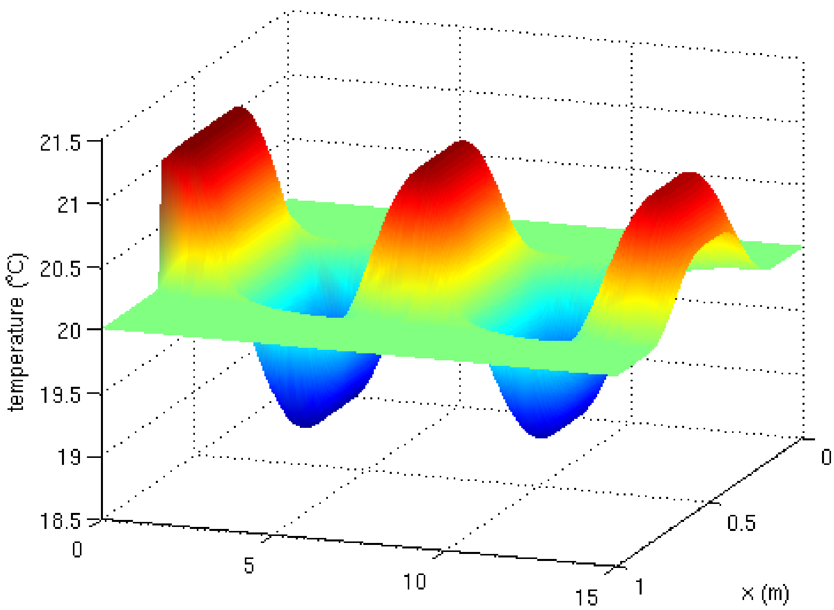

Fig. 6.8. Output of the plant, control applied with the controller R_1

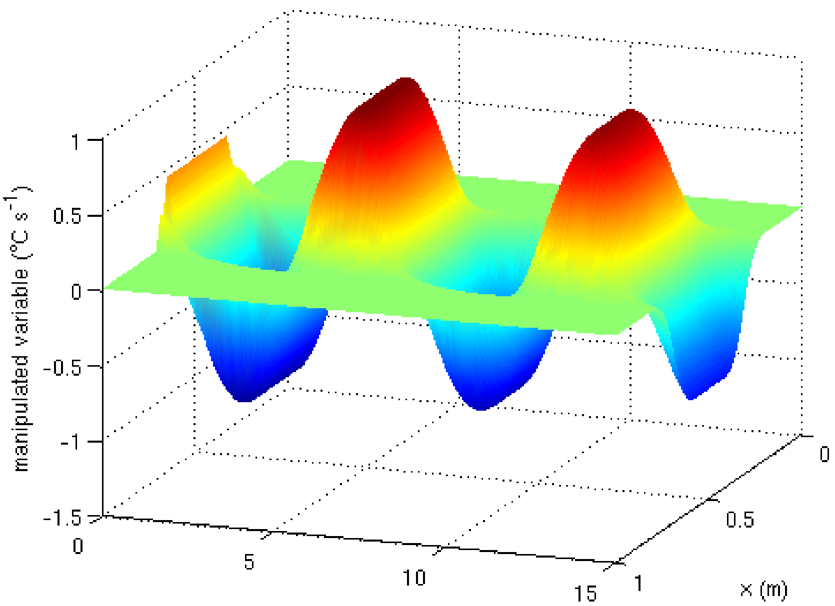

Fig. 6.9. Manipulated variable, control applied with the controller R_1

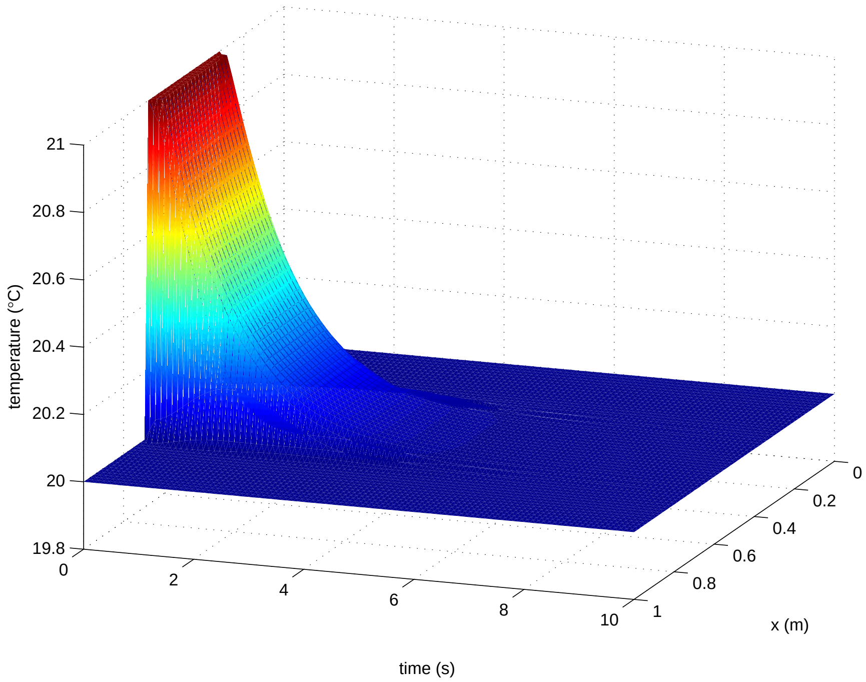

Fig. 6.10. Output of the plant, control applied with the controller R_2

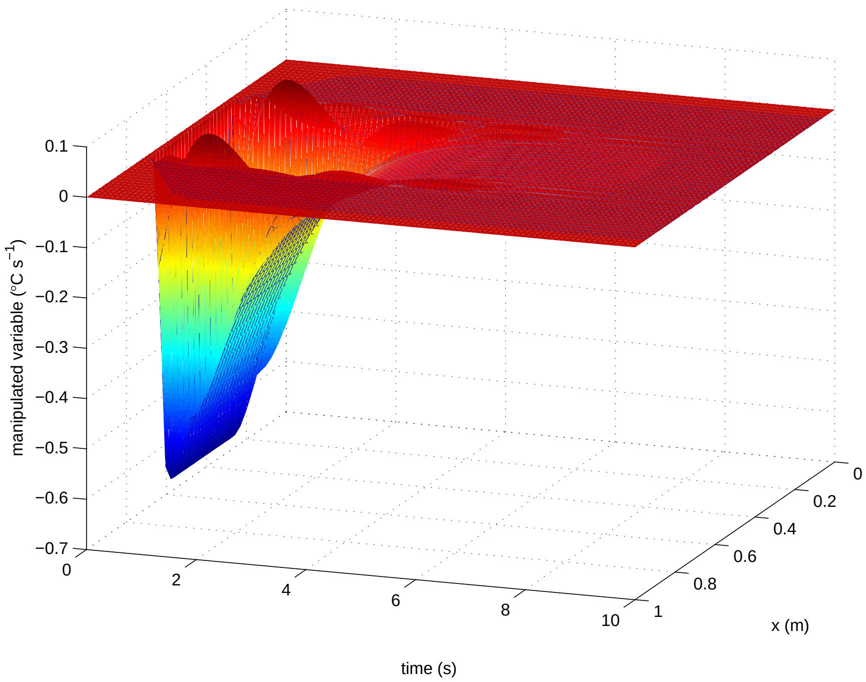

Fig. 6.11. Manipulated variable, control applied with the controller R_2

and the corresponding controller

$$R_2(s,w) = \frac{18633 - 2231\,s\,(w + w^{-1})}{15494 + 10000\,s + 5531\,(w + w^{-1})}. \tag{6.17}$$

The results with controller R_2 are shown in Figs. 6.10 and 6.11. □

6.3. LQG CONTROL

In this section, a simple procedure for design of LQG-optimal controller will be proposed. A controller optimal in the sense of minimising quadratic criterion

$$J = \sum_{k=0}^{\infty} \sum_{l_1=-\infty}^{\infty} \cdots \sum_{l_n=-\infty}^{\infty} \phi\, u^2(k, l_1, \ldots, l_n) + \psi\, y^2(k, l_1, \ldots, l_n) \tag{6.18}$$

where u and y denote input and output of a plant, respectively, will be designed.

Consider the control scheme of Fig. 6.12, where plant, controller and filter are respectively given by transfer functions

$$P = \frac{b}{a}, \qquad C = \frac{n}{m}, \qquad Q = \frac{c}{a}$$

and v and w are white noises. Let the polynomial a have the form

$$a = \sum_i \sum_{i_1} \cdots \sum_{i_n} a_{i,i_1,\ldots,i_n}\, z^i\, w_1^{i_1} \cdots w_n^{i_n}$$

and similarly for other polynomials. We suppose that the plant and the controller are causal in the sense of explicit difference scheme (see p. 13), i. e.,

$$b_{0,\cdot} = 0,$$
$$a_{0,0,\ldots,0} \neq 0, \qquad a_{0,i_1,\ldots,i_n} = 0, \quad i_1, \ldots, i_n \neq 0$$

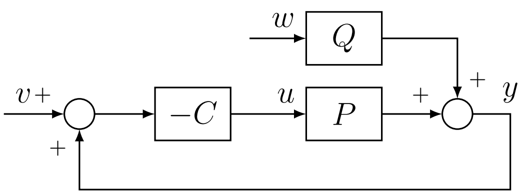

Fig. 6.12. LQG control scheme

and

$$n_{0,\cdot} = 0,$$
$$m_{0,0,\ldots,0} \neq 0, \qquad m_{0,i_1,\ldots,i_n} = 0, \quad i_1, \ldots, i_n \neq 0,$$

where a dot $(\cdot)$ means "for all indices". We also suppose that a, b have no zero in common, then Bézout identity

$$a\,m + b\,n = 1 \tag{6.19}$$

is satisfied.

Following theorem holds.

6.13. Theorem. Controller optimal in the sense of minimising criterion (6.18) is given by transfer function

$$C = \frac{n}{m}, \tag{6.20}$$

where

$$n(z, w_1, \ldots, w_n) = \frac{Y(z, w_1, \ldots, w_n)}{E(z, w_1, \ldots, w_n)\, D(z, w_1, \ldots, w_n)}, \tag{6.21}$$

$$m(z, w_1, \ldots, w_n) = \frac{X(z, w_1, \ldots, w_n)}{E(z, w_1, \ldots, w_n)\, D(z, w_1, \ldots, w_n)}, \tag{6.22}$$

where X and Y is solution of

$$z^p\, E^*\, X - Z\, b = z^p\, a^*\, \phi\, D \tag{6.23}$$
$$z^p\, E^*\, Y + Z\, a = z^p\, b^*\, \psi\, D \tag{6.24}$$

such that $Y_{0,\cdot} = 0$, where $p = \max(\deg a, \deg b, \deg E)$ and

$$a\, \phi_1\, a^* + c\, \psi_1\, c^* = D\, D^* \tag{6.25}$$
$$a\, \phi\, a^* + b\, \psi\, b^* = E\, E^* \tag{6.26}$$

where

$$D_{0,0,\ldots,0} \neq 0, \qquad D_{0,i_1,\ldots,i_n} = 0, \quad i_1, \ldots, i_n \neq 0$$

and

$$E_{0,0,\ldots,0} \neq 0, \qquad E_{0,i_1,\ldots,i_n} = 0, \quad i_1, \ldots, i_n \neq 0.$$

Minimal value of criterion (6.18) is

$$v_{\min} = v_1 + v_2 + v_3, \tag{6.27}$$

where

$$\begin{aligned} v_1 &= -\phi_1\,\psi \\ v_2 &= \left\langle \frac{D^*\,\phi\,D}{a^*\,a} - \frac{D^*\,\psi\,b\,b^*\,\psi\,D}{a^*\,E\,E^*\,a} \right\rangle \\ v_3 &= \left\langle \frac{Z^*\,Z}{E\,E^*} \right\rangle . \end{aligned}$$

Proof. The proof is similar to that of 1-D case [41]. Input and output of plant is respectively given by

$$u = -\frac{C}{1+C\,P}\,v - \frac{C\,Q}{1+C\,P}\,w = -a\,n\,v - n\,c\,w \tag{6.28}$$

$$y = -\frac{C\,P}{1+C\,P}\,v + \frac{Q}{1+C\,P}\,w = -b\,n\,v + m\,c\,w. \tag{6.29}$$

Using (6.28) a (6.29) and Parseval theorem, (6.18) becomes

$$J = \phi\,\langle a\,n\,\phi_1\,n^*\,a^* + n\,c\,\psi_1\,c^*\,n^*\rangle + \psi\,\langle b\,n\,\phi_1\,n^*\,b^* + m\,c\,\psi_1\,c^*\,m^*\rangle$$

and using (6.19) and (6.25)

$$J = \phi\,\langle n\,D\,D^*\,n^*\rangle + \psi\,\langle -\phi_1 + \phi_1\,n^*\,b^* + b\,n\,\phi_1 + m\,D\,D^*\,m^*\rangle\,.$$

Let $J = v_1 + v_4$ and

$$\begin{aligned} v_1 &= -\phi_1\,\psi \\ v_4 &= \left\langle n\,D\,\phi\,D^*\,n^* + \frac{1-b\,n}{a} D\,\psi\,D^* \frac{1-n^*\,b^*}{a^*} \right\rangle . \end{aligned}$$

where we used (6.19) and the fact that $\langle \phi_1\,n^*\,b^*\rangle = 0$ and $\langle b\,n\,\phi_1\rangle = 0$.

Next manipulations give

$$
\begin{aligned}
v_4 &= \Big\langle n\,D\,\phi\,D^*\,n^* + \frac{1}{a}D\,\psi\,D^*\frac{1}{a^*} - \frac{1}{a}D\,\psi\,D^*\frac{n^*\,b^*}{a^*} \\
&\qquad\qquad - \frac{b\,n}{a}D\,\psi\,D^*\frac{1}{a^*} + \frac{b\,n}{a}D\,\psi\,D^*\frac{n^*\,b^*}{a^*}\Big\rangle \\
&= \Big\langle \frac{1}{a}D\,\psi\,D^*\frac{1}{a^*} - \frac{1}{a}D\,\psi\,D^*\frac{n^*\,b^*}{a^*} - \frac{b\,n}{a}D\,\psi\,D^*\frac{1}{a^*}\Big\rangle \\
&\qquad\qquad + \Big\langle \frac{a}{a}n\,D\,\phi\,D^*\,n^*\frac{a^*}{a^*} + \frac{b\,n}{a}D\,\psi\,D^*\frac{n^*\,b^*}{a^*}\Big\rangle \\
&= \Big\langle \frac{1}{a}D\,\psi\,D^*\frac{1}{a^*} - \frac{1}{a}D\,\psi\,D^*\frac{n^*\,b^*}{a^*} - \frac{b\,n}{a}D\,\psi\,D^*\frac{1}{a^*}\Big\rangle \\
&\qquad\qquad + \Big\langle \frac{1}{a}n\,D\,(a\,\phi\,a^* + b\,\psi\,b^*)\,D^*\,n^*\frac{1}{a^*}\Big\rangle \\
&= \Big\langle \frac{1}{a}D\,\psi\,D^*\frac{1}{a^*} - \frac{1}{a}D\,\psi\,D^*\frac{n^*\,b^*}{a^*} \\
&\qquad\qquad - \frac{b\,n}{a}D\,\psi\,D^*\frac{1}{a^*} + \frac{1}{a}n\,D\,E\,E^*\,D^*\,n^*\frac{1}{a^*}\Big\rangle.
\end{aligned}
$$

and v_4 becomes

$$v_4 = v_2 + v_5, \tag{6.30}$$

where

$$v_2 = \Big\langle \frac{1}{a}D\,\psi\,D^*\frac{1}{a^*} - \frac{1}{E^*\,a}D\,\psi\,b^*\,b\,\psi\,D^*\frac{1}{a^*\,E}\Big\rangle \tag{6.31}$$

$$v_5 = \Big\langle \Big(\frac{1}{a}n\,D\,E - \frac{1}{E^*\,a}D\,\psi\,b^*\Big)\Big(\frac{1}{a}n\,D\,E - \frac{1}{E^*\,a}D\,\psi\,b^*\Big)^*\Big\rangle. \tag{6.32}$$

Using (6.24),

$$\frac{D\,\psi\,b^*}{E^*\,a} = \frac{Y}{a} + \frac{Z}{z^p\,E^*}.$$

Let

$$V = \frac{n\,D\,E}{a} - \frac{Y}{a}.$$

We get

$$v_5 = \Big\langle \Big(V - \frac{Z}{z^p\,E^*}\Big)\Big(V - \frac{Z}{z^p\,E^*}\Big)^*\Big\rangle$$

For D, E, a, n and Y satisfying the assumptions made above, cross-terms

$$\left\langle \frac{Z^* V}{(z^p E^*)^*} \right\rangle = \left\langle \frac{Z V^*}{(z^p E^*)} \right\rangle = 0$$

and

$$v_5 = v_3 + v_6,$$

where

$$v_3 = \left\langle \frac{Z^* Z}{E E^*} \right\rangle, \qquad v_6 = \langle V^* V \rangle .$$

Value of criterion is

$$v = v_1 + v_2 + v_3 + v_6.$$

Only v_6 depends on a controller. Putting $v_6 = 0$ gives

$$0 = V = \frac{n E D}{a} - \frac{Y}{a} = \frac{1}{a}(n E D - Y),$$

so,

$$n = \frac{Y}{D E}$$

From

$$\begin{aligned} z^p E^* X - Z b &= z^p a^* \phi D \\ z^p E^* Y + Z a &= z^p b^* \psi D \end{aligned}$$

then

$$m = \frac{X}{D E}$$

follows. □

The design of LQG controller consists of the following essential operations.

6.14. Algorithm (LQG controller). Input: *polynomials* a*,* b *describing a plant, weights* $\Psi, \Phi, \Psi_1, \Phi_1$. Output: *LQG controller transfer function* C.

1. Solve spectral factorisations (6.25) and (6.26).
2. Solve Diophantine equations (6.23) and (6.24).

3. Substitute the solution into (6.21) and (6.22) to get n and m.

4. Substitute n and m into (6.20) to get the transfer function of controller. □

An example on LQG control design follows.

6.15. Example. Consider a heat conduction in a rod with array of temperature sensors and heaters described by (4.13). Fig. 3.5 and Fig. 3.6 give the initial condition and the response of the uncontrolled system, respectively. The above described methods was used to design LQG controllers for $\phi_1 = 1$ and $\psi_1 = 1$. Figs. 6.13, 6.14 and 6.15, 6.16 show responses to initial condition of closed-loop system and the manipulated variable with LQG controller for weights $\phi = 1$, $\psi = 1$ and $\phi = 1$, $\psi = 100$, respectively.

We will show how to design the LQG controller in Matlab environment using MuPAD and polmat. In MuPAD, load the polmat library. Define the plant and filter

```
package("polmat"):
a:=subs(1-T*K/h^2*z*(w+1/w)-(1-2*T*K/h^2)*z,
   T=1, K=230/2700/900, h=1/59):
b:=z: c:=b:
```

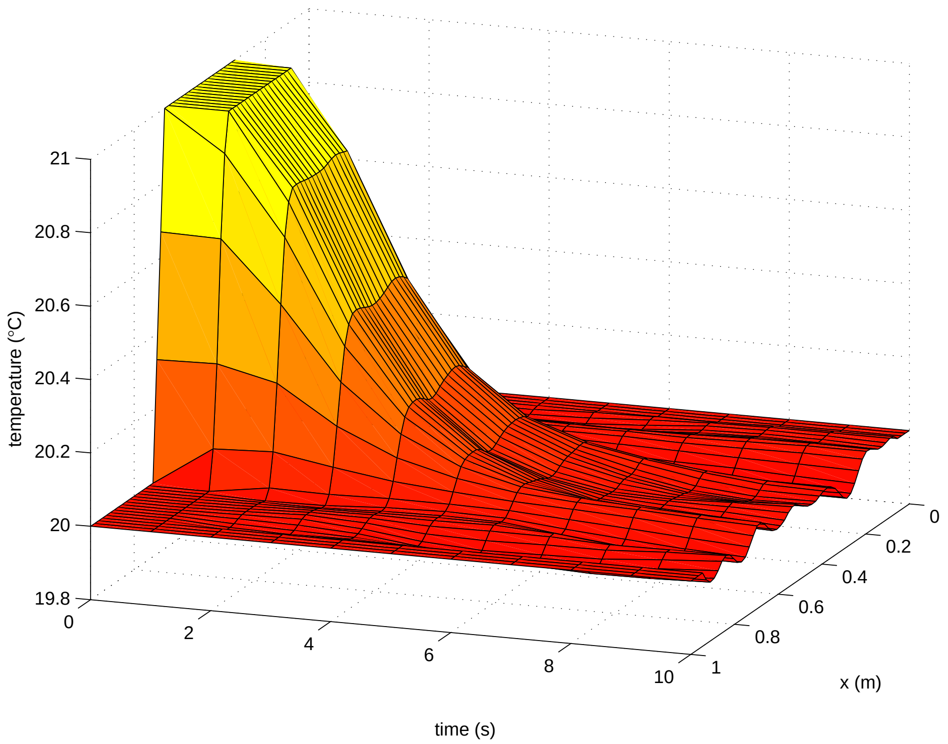

Fig. 6.13. LQG control, $\phi_1 = 1$, $\psi_1 = 1$, $\phi = 1$, $\psi = 1$, output of plant

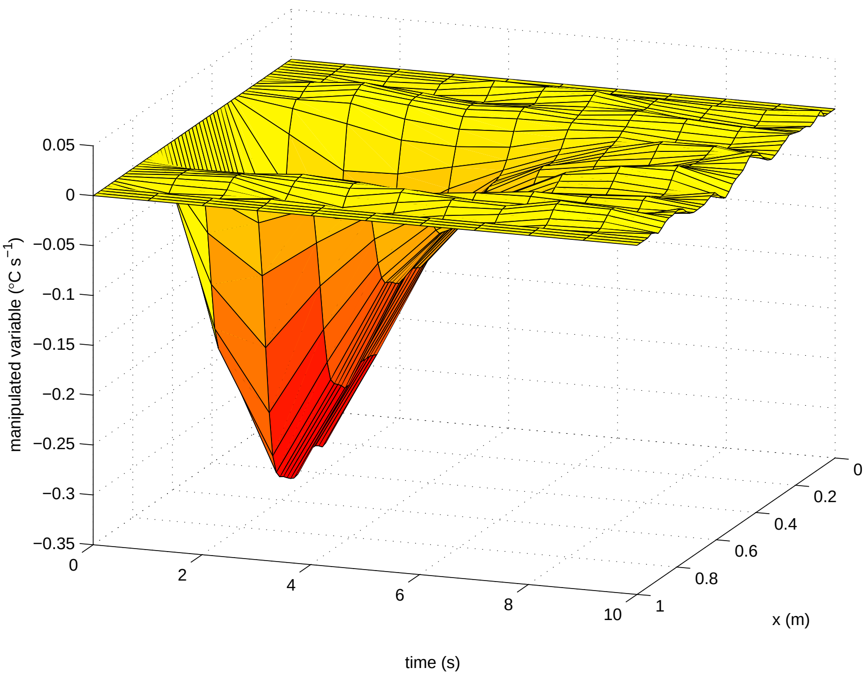

Fig. 6.14. LQG control, $\phi_1 = 1$, $\psi_1 = 1$, $\phi = 1$, $\psi = 1$, manipulated variable

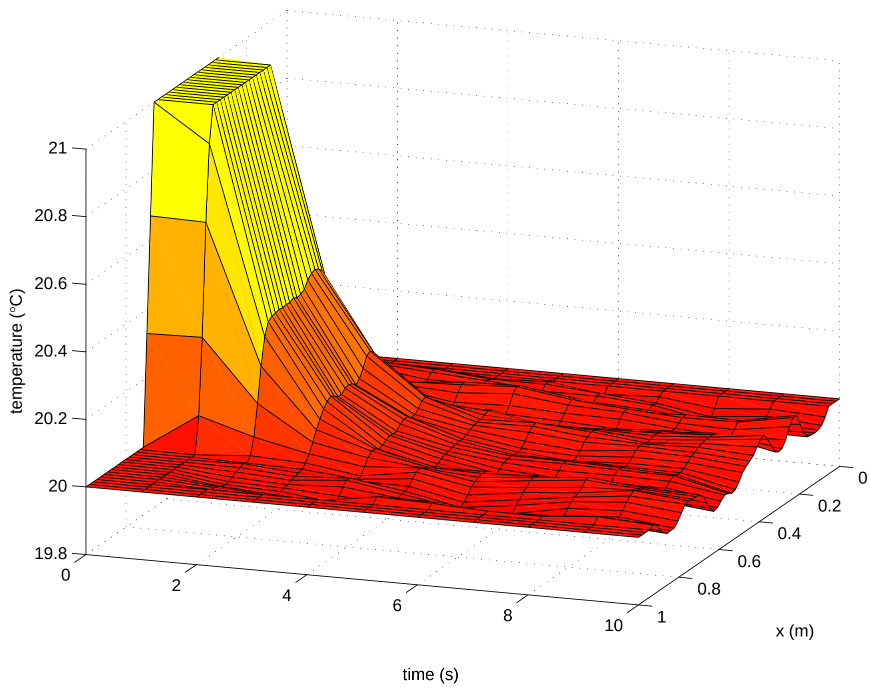

Fig. 6.15. LQG control, $\phi_1 = 1$, $\psi_1 = 1$, $\phi = 1$, $\psi = 100$, output of plant

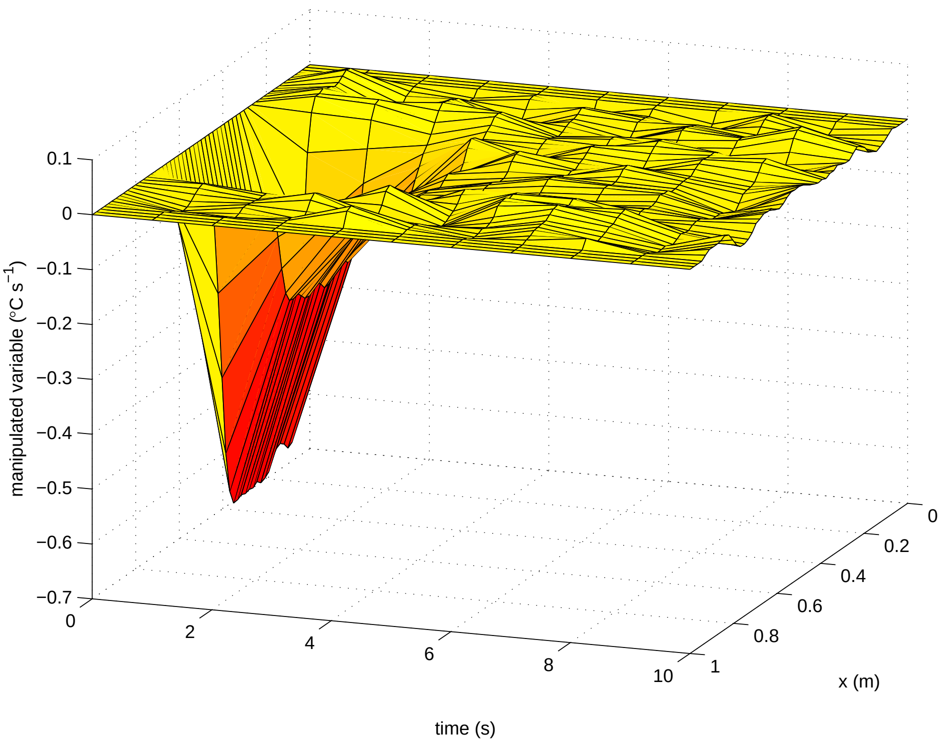

Fig. 6.16. LQG control, $\phi_1 = 1$, $\psi_1 = 1$, $\phi = 1$, $\psi = 100$, manipulated variable

and constants ϕ, ψ, ϕ_1 and ψ_1

```
Phi:=1: Psi:=100:
Phi_1:=1: Psi_1:=1:
```

Build two spectral factorisations

```
f:=a*Phi*subs(a, z=1/z) + b*Psi*subs(b, z=1/z)
f1:=a*Phi_1*subs(a, z=1/z) + c*Psi_1*subs(c, z=1/z)
```

They must written in the form of list of coefficients to be solved using Algorithm 7.4. This form can be generated in MuPAD as follows

```
d_f1_z:=polmat::ldeg(f, z): d_f1_w:=polmat::ldeg(f, w):
d_f_z:=polmat::ldeg(f, z): d_f_w:=polmat::ldeg(f, w):
(m1[i,j]:=polmat::coeff(polmat::coeff(f1, z, abs(i)), w,
          abs(j))) $i=-d_f1_z..d_f1_z $j=-d_f1_w..d_f1_w:
(m[i,j]:=polmat::coeff(polmat::coeff(f, z, abs(i)), w,
          abs(j))) $i=-d_f_z..d_f_z $j=-d_f_w..d_f_w:
M1:=matrix(1+2*d_f1_z,
          1+2*d_f1_w,(i,j)->m1[i-1-d_f1_z,j-1-d_f1_w]):
M:=matrix(1+2*d_f_z, 1+2*d_f_w,(i,j)->m[i-1-d_f_z,j-1-d_f_w]):
```

Lists can be generated by

```
polmat::polmat2poltbx(M1)
polmat::polmat2poltbx(M)
```

Now, let us switch on Matlab. The spectral factor can be computed by the following code, see Sec. 7.2 for details.

```
f=[0, -80063/243000, -41437/121500, -80063/243000, 0; ...
   6410083969/59049000000, 3317570531/14762250000, ...
   22964377969/9841500000, 3317570531/14762250000, ...
   6410083969/59049000000; ...
   0, -80063/243000, -41437/121500, -80063/243000, 0];

[m,d]=size(f);
n=2^7;
p=[f((m+1)/2:m,:); zeros(n-m,d); f(1:(m+1)/2-1,:)];
p=[p(:,(d+1)/2:d), zeros(n, n-d), p(:,1:(d+1)/2-1)];
P=fft2(p,n,n);
b=ifft2(log(P),n,n);
bp=b(1:(n+1)/2, :); bp(1,:)=bp(1,:)/2;
g=ifft2(exp(fft2(bp)));
v=g(1:(m+1)/2, 1:(d+1)/2)
```

Let us continue in MuPAD. Rewrite the spectral factors

```
DD:=matrix(2, 3, [15/10, 0, 0, -2/10, -2/10, 0]);
EE:=matrix(2, 3, [101/10, 0, 0, -34/1000, -33/1000, 0]);
[l1,l2]:=linalg::matdim(DD)
vD:=(linalg::transpose(matrix([z^i $i=0..l1-1]))*DD*matrix([1,
    w^i+1/w^i $i=1..l2-1]))[1]
vE:=(linalg::transpose(matrix([z^i $i=0..l1-1]))*EE*matrix([1,
    w^i+1/w^i $i=1..l2-1]))[1]
```

In `vD` and `vE` we have now spectral factors in the MuPAD data type *expression*. One can make sure that to obtain the controller transfer function we can solve Diophantine equation $a\,X + b\,Y = E\,D$ in place of Eqs. (6.23) and (6.24). We take the solution with $Y_{0,\cdot}$ by two last lines.

```
[x,y]:=polmat::axbyc(a, b, vE*vD, z)
solve(polmat::coeff(y, z, 0), t)
[X,Y]:=expand(expr(subs([x,y], t=%[2])))
```

□

6.4. $\mathcal{H}_2$-OPTIMAL CONTROLLER

In this section, parametrisation of all stabilising controllers will be derived for the plant described by (4.12). Among these stabilising controllers, a controller minimising $\mathcal{H}_2$ norm of the closed-loop system transfer function will be selected.

The following theorem gives a set of all stabilising controllers. It has origin in [41]. For multidimensional systems it appeared in [59].

6.16. Theorem (Kučera-Youla) [59]. Let $\hat{p}, \hat{q} \in \mathbb{R}[z, w]$ be any polynomials such that

$$a\,\hat{p} + b\,\hat{q} = s$$

is stable. Then the plant, free of hidden modes, with factor coprime transfer function

$$P = \frac{b}{a}$$

is stabilisable and the set of all stabilising controllers is given by

$$C = \frac{q}{p} = \frac{\hat{q}\,r - a\,t}{\hat{p}\,r + b\,t} \tag{6.33}$$

with all possible common factors cancelled, where r is an arbitrary stable n-D polynomial and the polynomial t is arbitrary, $r, t \in \mathbb{R}[z, w]$. □

The stabilising controller minimising $\mathcal{H}_2$ norm will be derived from the set of all stabilising controllers (6.33). The task is to find a controller that internally stabilises the control system of Fig. 6.17 and simultaneously minimises the effect of the signal v on the output y in the sense of minimising the $\mathcal{H}_2$ norm of the transfer function

$$H = \frac{P}{1 + P\,C}, \tag{6.34}$$

which is defined by

$$\|H\|^2 = \frac{1}{4\,\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| H\left(e^{j\,\omega}, e^{j\,\omega_1}\right) \right|^2 \,\mathrm{d}\omega_1 \,\mathrm{d}\omega.$$

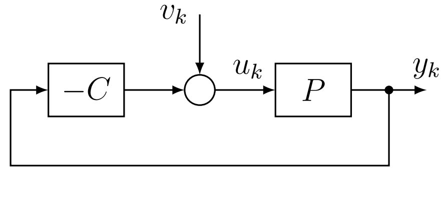

Fig. 6.17. Closed-loop system

By modification of (6.34) we have

$$H = \frac{p\,b}{a\,p + b\,q} = \frac{(\hat{p}\,r + b\,t)b}{r\,s} = \Upsilon + \Phi\,W,$$

where Υ and Φ are some stable rational functions and $W = \frac{t}{r}$. Consider the inner-outer factorisation [14] of Φ as $\Phi = \Phi_\mathrm{i}\,\Phi_\mathrm{o}$, where Φ_i has unit magnitude on the unit circle and Φ_o has no zeros in $|z| \leq 1$. With this factorisation,

$$\|\Upsilon - \Phi\,W\|_2 = \|\Upsilon - \Phi_\mathrm{i}\,\Phi_\mathrm{o}\,W\|_2 = \left\|\Phi_\mathrm{i}\left(\tfrac{\Upsilon}{\Phi_\mathrm{i}} - \Phi_\mathrm{o}\,W\right)\right\|_2 = \left\|\tfrac{\Upsilon}{\Phi_\mathrm{i}} - \Phi_\mathrm{o}\,W\right\|_2.$$

Now, decompose $\frac{\Upsilon}{\Phi_\mathrm{i}}$ as

$$\frac{\Upsilon}{\Phi_\mathrm{i}} = \left\{\frac{\Upsilon}{\Phi_\mathrm{i}}\right\}_+ + \left\{\frac{\Upsilon}{\Phi_\mathrm{i}}\right\}_-,$$

where $\{\cdot\}_+$ is analytic in $|z| \leq 1$ and $\{\cdot\}_-$ is strictly proper and analytic in $|z| \geq 1$. With this decomposition,

$$\begin{aligned} \left\|\tfrac{\Upsilon}{\Phi_\mathrm{i}} - \Phi_\mathrm{o}\,W\right\|_2^2 &= \left\|\left\{\tfrac{\Upsilon}{\Phi_\mathrm{i}}\right\}_+ + \left\{\tfrac{\Upsilon}{\Phi_\mathrm{i}}\right\}_- - \Phi_\mathrm{o}\,W\right\|_2^2 \\ &= \left\|\left\{\tfrac{\Upsilon}{\Phi_\mathrm{i}}\right\}_+\right\|_2^2 + \left\|\left\{\tfrac{\Upsilon}{\Phi_\mathrm{i}}\right\}_- - \Phi_\mathrm{o}\,W\right\|_2^2. \end{aligned}$$

The last expression is a complete square whose first part is independent of W. Hence, the minimising parameter is

$$W = \frac{\left\{\frac{\Upsilon}{\Phi_\mathrm{i}}\right\}_+}{\Phi_\mathrm{o}} \qquad (6.35)$$

If W is proper and stable, it defines the unique optimal controller. If not, no optimal controller exists. See [42] for details on the above steps.

The above steps can be summarised as follows.

6.17. Algorithm ($\mathcal{H}_2$-optimal controller). Input: *polynomials a, b describing a plant*. Output: *$\mathcal{H}_2$-optimal controller transfer function C*.

1. Solve $a\,\hat{p} + b\,\hat{q} = s$ for $\hat{p}$, $\hat{q}$.
2. Let $\Upsilon = \frac{\hat{p}}{s}$, $\Phi = \frac{b^2}{s}$.
3. Factorise $\Phi = \Phi_\mathrm{i}\,\Phi_\mathrm{o}$. Let Φ_i have unit magnitude on the unit circle. Let Φ_o have no zeros in $|z| \leq 1$.

4. Decompose $\dfrac{\Upsilon}{\Phi_{\mathrm{i}}} = \left\{\dfrac{\Upsilon}{\Phi_{\mathrm{i}}}\right\}_+ + \left\{\dfrac{\Upsilon}{\Phi_{\mathrm{i}}}\right\}_-$. Let $\left\{\dfrac{\Upsilon}{\Phi_{\mathrm{i}}}\right\}_+$ be analytic in $|z| \leq 1$. Let $\left\{\dfrac{\Upsilon}{\Phi_{\mathrm{i}}}\right\}_-$ be strictly proper and analytic in $|z| \geq 1$.

5. $W = \dfrac{\left\{\frac{\Upsilon}{\Phi_{\mathrm{i}}}\right\}_+}{\Phi_{\mathrm{o}}}$.

6. $C = \dfrac{\hat{q} - a\,W}{\hat{p} + b\,W}$. □

In the following example, $\mathcal{H}_2$-optimal controller for the system of heat conduction in a rod will be designed. The $\mathcal{H}_2$-optimal controller for a thin flexible plate of Example 3.2 was scope of [4]. Note that steps 3 and 4 in the above algorithm are not hard to compute for the particular systems considered here since the degrees of the polynomials Υ and Φ do not exceed 2.

6.18. Example. Consider a plant (4.12). The equation $a\,\hat{p} + b\,\hat{q} = 1$ is solvable and a solution is

$$\hat{p} = 1, \quad \hat{q} = 1 - 2\,\frac{T}{h^2} + \frac{T}{h^2}\left(w + w^{-1}\right).$$

Hence the set of all stabilising controllers is given by

$$R = \frac{\left[1 - 2\frac{T}{h^2} + \frac{T}{h^2}\left(w + w^{-1}\right)\right](r + z\,t) - t}{r + z\,t}, \tag{6.36}$$

where r is a stable 2-D polynomial and t is an arbitrary polynomial.

We have $\Upsilon = z$, $\Phi = z^2$, so, $\Phi_{\mathrm{i}} = z^2$, $\Phi_{\mathrm{o}} = 1$ and

$$\frac{\Upsilon}{\Phi_{\mathrm{i}}} = \frac{z}{z^2} = \left\{\frac{\Upsilon}{\Phi_{\mathrm{i}}}\right\}_+ + \left\{\frac{\Upsilon}{\Phi_{\mathrm{i}}}\right\}_- = 0 + \frac{1}{z}$$

and (6.35) gives $W = \frac{0}{1} = 0$. Finally, we obtain transfer function of stabilising controller minimising $\mathcal{H}_2$ norm by substitution in (6.36) as

$$R = 1 - 2\,\frac{T}{h^2} + \frac{T}{h^2}\left(w + w^{-1}\right).$$

For a numerical simulation consider the case of a rod with transfer function (4.13). Let signal v_k be given in Fig. 3.5 for $k = 0$ and zero for $k > 0$. Fig. 6.18 and 6.19 correspond to the case when the controller minimising the $\mathcal{H}_2$ norm is used. □

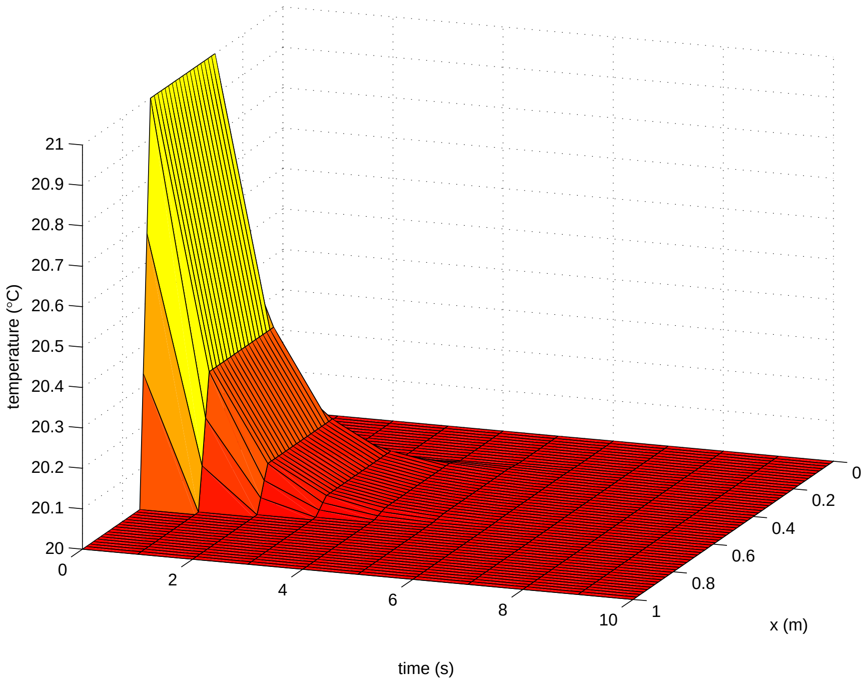

Fig. 6.18. Temperature variation with the $\mathcal{H}_2$-optimal controller applied

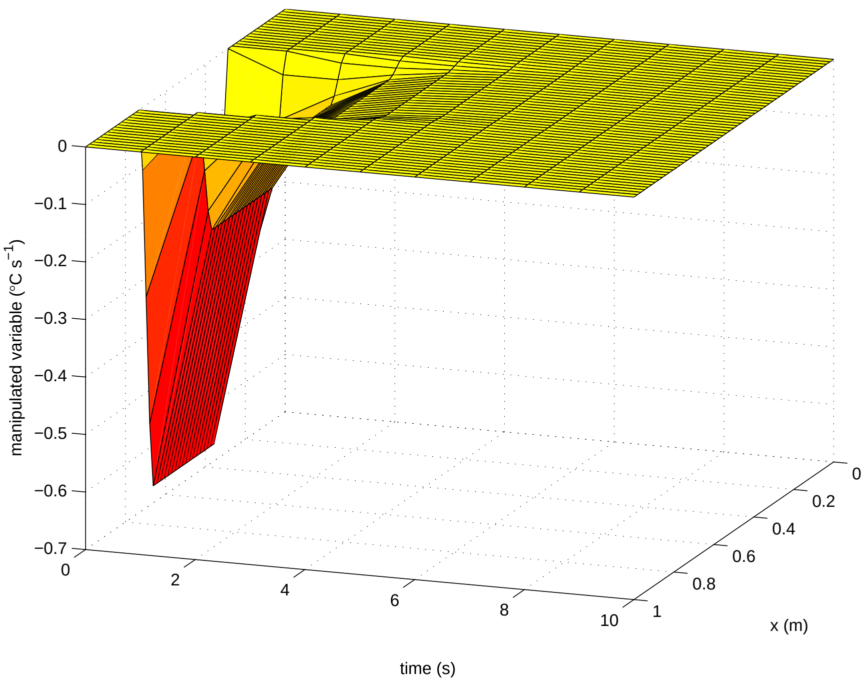

Fig. 6.19. Manipulated variable with the $\mathcal{H}_2$-optimal controller applied

6.5. DEAD-BEAT CONTROL

We will design so-called *dead-beat* control. Consider scheme of Fig. 6.20. The task is to design a controller which drives control error e_k to zero in a least number of steps and, furthermore, guarantees that output y_k tracks reference r_k of type step. Suppose that controller contains unit delay, in other words, computing new controller output takes a time Δt. Let a controller be a series connection of two components. The first one, with transfer function R, is subject to design and the second one is fixed and guarantees asymptotic properties of the controller. In 1-D systems, this task was solved in e. g. [49].

Consider a plant and a controller given by transfer function

$$P = \frac{b}{a}, \qquad R = \frac{y}{x},$$

respectively. When in design, we will consider the second controller component to be part of a plant, i. e.

$$P_{\mathrm{I}} = \frac{z}{1-z} P.$$

For the closed loop it reads

$$e = \frac{1}{1 + P_{\mathrm{I}}\, R}\, r = \frac{(1-z)\, a\, x}{(1-z)\, a\, x + z\, b\, y}\, r. \tag{6.37}$$

We suppose reference to be step

$$r(z,w) = \frac{1}{1-z}\, r'(w),$$

so, (6.37) becomes

$$e = \frac{a\, x}{(1-z)\, a\, x + z\, b\, y}\, r'. \tag{6.38}$$

Since we need (6.38) to be a polynomial in z, we have

$$(1-z)\, a\, x + z\, b\, y = s, \tag{6.39}$$

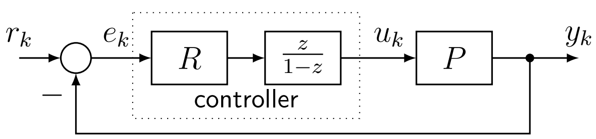

Fig. 6.20. Closed-loop system with the dead-beat controller [49]

where s is a constant polynomial in the ring $\mathbb{R}[w][z]$, i. e., independent on z. Now, (6.38) becomes

$$e = a\,x\,\frac{r'}{s}.$$

Set of all dead-beat controllers (guaranteeing reference tracking) is given by

$$R = \frac{\hat{y}\,t - (1-z)\,a\,v}{\hat{x}\,t + z\,b\,v}, \tag{6.40}$$

where $t, w \in \mathbb{R}[w][z]$, t is arbitrary stable and v is arbitrary. Control error is

$$e = a\left(\hat{x}\,t + b\,v\right)\frac{r'}{s}.$$

The parameters t, v determine a particular controller of the set (6.40). Since our objective is to reach a least minimum of steps, the parameters t, v must be choose to minimise the degree of polynomial $\hat{x}\,t + b\,v$. In other words, we must solve (6.39) and take the solution minimising the degree of x.

The polynomials r' and s describe reference in spatial coordinates and are function only of w. The method does not require knowing r' to design dead-beat control.

The above described method can be summarised as follows.

6.19. Algorithm (Dead-beat control). Input: *polynomials a, b describing a plant.* Output: *transfer function R of dead-beat controller.*

1. Choose $s(w)$ that $(1-z)\,a\,x + z\,b\,y = s$ is solvable.
2. Solve $(1-z)\,a\,x + z\,b\,y = s$ for x, y. Take the solution minimising the degree of x in the variable z.
3. Dead-beat controller is given by $R = \frac{y}{x}$. The control error goes to zero in number of steps equal to the degree of $a\,x$ in the variable z. □

6.20. Example. Consider a heat conduction in a rod given by (4.13). The dead-beat controller was designed using the above method. The degree of e resulted 3. The simulations are shown in Figs. 6.21–6.23. One can see that signal e_k goes to zero in 3 steps. □

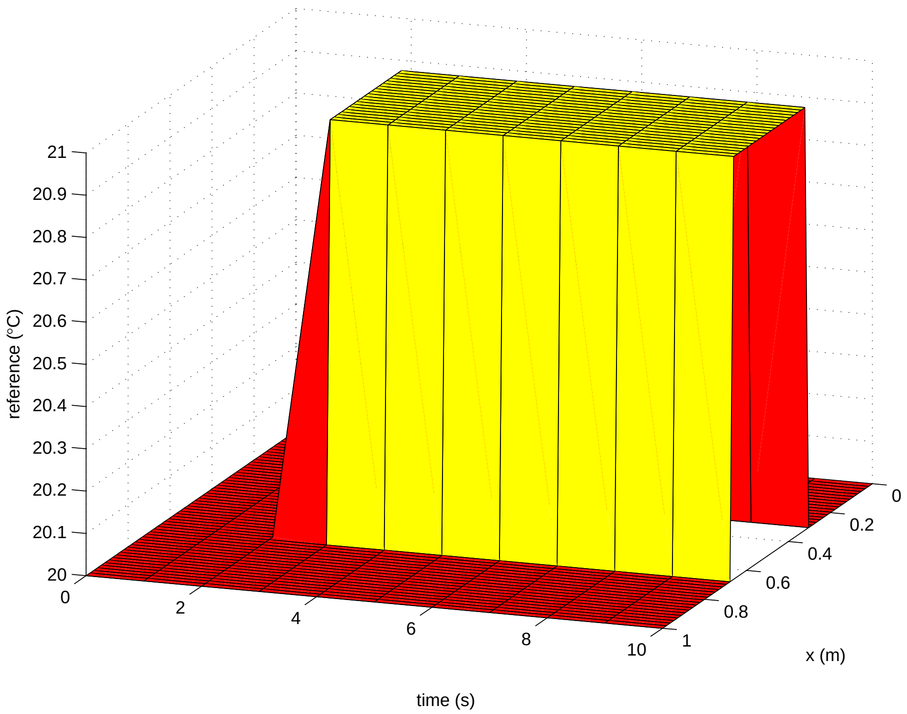

Fig. 6.21. Reference

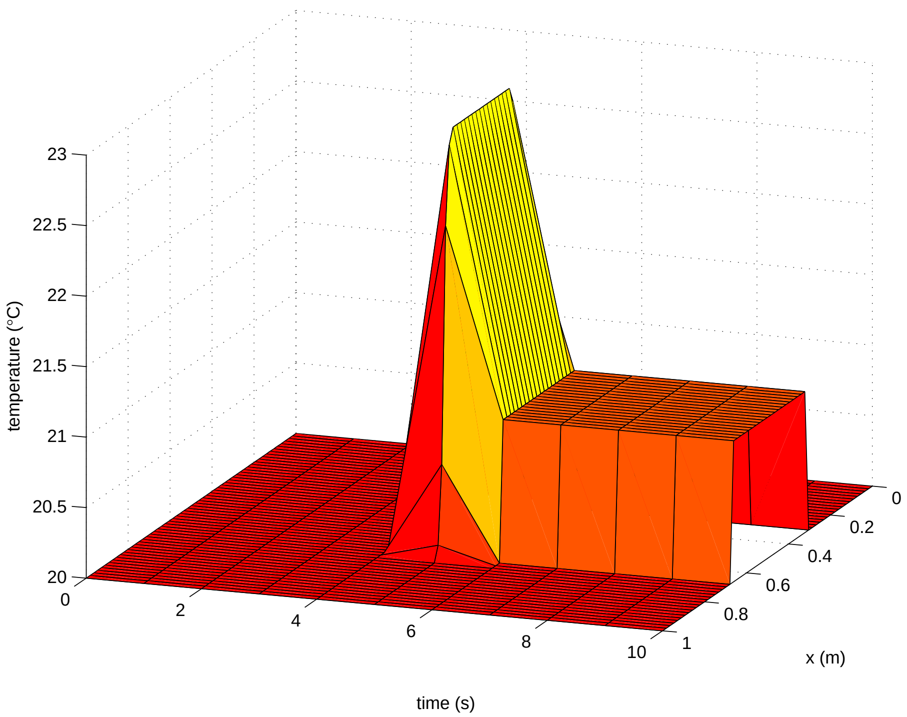

Fig. 6.22. Output

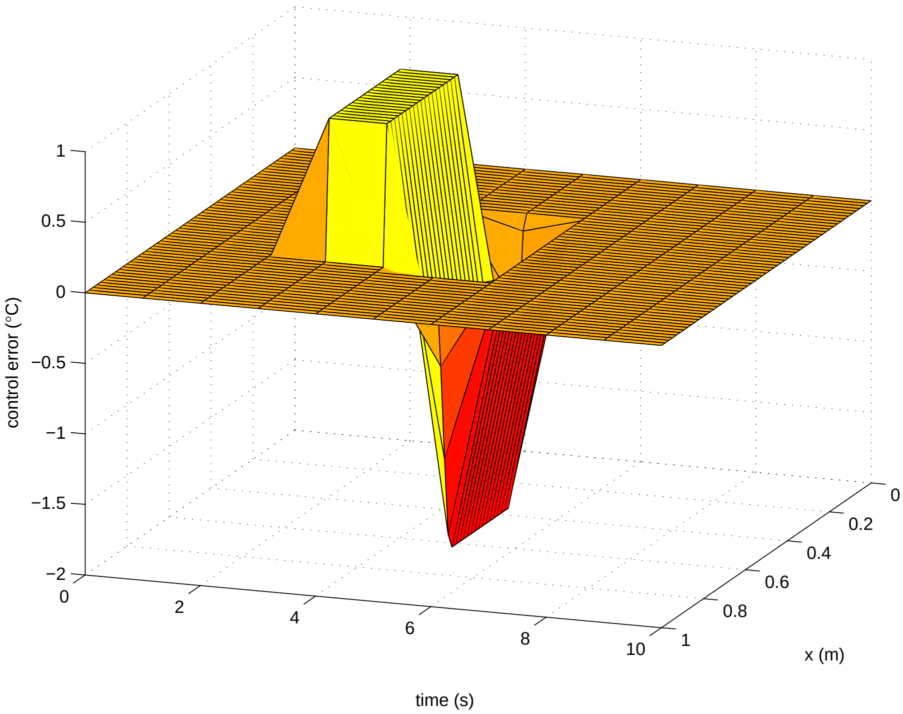

Fig. 6.23. Control error

CHAPTER 7

Tools

This chapter treats of multivariate linear polynomial equations and spectral factorisation problems.

7.1. LINEAR MULTIVARIATE POLYNOMIAL EQUATIONS

As we saw in the previous chapters, linear equations with polynomials (also called Diophantine equations) play a central role in the control design via algebraic methods. In this section, we concern in the linear equation with multivariate polynomials introduced before in (6.2). Let $a, b, c \in \mathbb{R}[z, w]$ be given polynomials two-sided in the variable w and one-sided in the variable z, and x and y be unknown elements of the same ring. A condition of solvability will be given.

Note that concept of coprimeness of one-variable polynomials was extended to the multivariate case in [73]. Concerning solvability of linear equations, multidimensional case was scope of e. g. [58, 59].

Let us begin with a definition of *ideal*, which can be find in e. g. [45, 74]. A subset $I \subset \mathbb{R}[z, w]$ is called *ideal* if two following conditions hold:

1. $a, b \in I \Rightarrow a - b \in I$;
2. $a \in I, f \in \mathbb{R}[z, w] \Rightarrow f\,a \in I$.

Let us have a subset $M \subset \mathbb{R}[z, w]$. Ideal $I(M)$ generated by M consists of finite $\mathbb{R}$-linear combinations of elements of M

$$\lambda_1\, m_1 + \cdots \lambda_r\, m_r,$$

where $\lambda_i \in \mathbb{R}[z, w]$ and $m_i \in M$.

Condition of solvability of (6.2) is established by *Hilbert's nullstellensatz*. Hilbert's nullstellensatz can be expressed in various ways. Here we use the interpretation of [74].

7.1. Theorem (Hilbert's nullstellensatz)**.** Let

$$f, f_1, \dots, f_r \in \mathbb{R}[z, w]$$

and f vanishes in all common zeros of $f_1, \dots, f_r$. Then for some $q \in \mathbb{N}$ polynomial f^q belongs to ideal generated by polynomials $f_1, \dots, f_r$, i. e.

$$f^q = g_1 f_1 + \dots + g_r f_r$$

for some $g_1, \dots, g_r \in \mathbb{R}[z, w]$.
Proof. See [74]. □

The below corollary states solvability condition of (6.2).

7.2. Corollary. The equation

$$a\,x + b\,y = c,$$

where $a, b, c \in \mathbb{R}[z, w]$ are given polynomials and x, y are unknown elements of the same ring, is solvable if and only if a, b vanish implies c vanishes. □

The next corollary (see also [59]) deals with the case, when $c = 1$.

7.3. Corollary. Bézout's identity

$$a\,x + b\,y = 1,$$

where $a, b \in \mathbb{R}[z, w]$ are given polynomials and x, y are unknown elements of the same ring, is solvable if and only if a, b have no zero in common. □

A possible way how to solve multivariate polynomial linear equations is to consider the polynomials to be element of $\mathbb{R}[w][z]$. Algorithms by Kučera [41] for univariate polynomial linear equations can then be used.

7.2. MULTIVARIATE POLYNOMIAL SPECTRAL FACTORISATION

The spectral factorisation as a mathematical tool was invented by Wiener in 1950's. Since that time, both univariate and multivariate spectral factorisation have been subject of many papers, see e. g. [31, 22, 21] and references herein. In this section, two-variable polynomial spectral factorisation is solved. The extension of the proposed algorithm to the n-variable case is straightforward.

Suppose we are given a real two-variable two-sided polynomial f of the form

$$f = \sum_{k=0}^{m} \sum_{l=0}^{s} f_{k,l} \left(z^k + z^{-k}\right) \left(w^l + w^{-l}\right), \qquad 0 \leq m, \quad 0 \leq s$$

positive on the unit bicircle,

$$0 < f < \infty \qquad \forall\, |z| = 1,\ |w| = 1.$$

Note that the requirement of positiveness is related to [75]. A polynomial $g(z, w)$ is the spectral factor of f if and only if

$$f(z, w) = g(z, w) \cdot g(z^{-1}, w),$$

$$g(z, w) \neq 0 \qquad \forall\, |z| \leq 1,\ |w| = 1.$$

It is well-known fact that, unlike univariate case, in general, it is not possible to find polynomial spectral factor of finite form. However, there exist a factorisation that has infinite number of terms, and it can be approximated by finite polynomial. See e. g. [17] for details.

There are several methods for computation of multivariate polynomial spectral factorisation, see e. g. [17, 21, 40, 51, 11, 52]. In [17, 21, 11], so-called *cepstral method* is discussed. Roughly speaking, its basic principle consists in computing cepstrum $\hat{f} = \log(f)$ and taking its causal part $\hat{f}_+$. Spectral factor is then given by $g = \exp(\hat{f}_+)$. Numerical implementation with use of fast Fourier transform (FFT) leads to very efficient algorithm. Clearly, its accuracy depends significantly on number of interpolation points used in FFT. If no finite spectral factor exists or its degree is too high, this algorithm truncates the resulting sequence and returns directly a finite-order spectral factor. It can be shown [21] that if truncated spectral factor is of sufficiently high order, stability is preserved. It is also possible to combine this algorithm with optimisation routine to produce an approximation of spectral factor.

For a clarity, the algorithm will be summarised for systems with one temporal and one spatial coordinate. Let the polynomial g have the form

$$g = \sum_{k=0}^{\hat{m}} \sum_{l=0}^{\hat{s}} g_{k,l}\, z^k \left(w^l + w^{-l}\right), \qquad 0 \leq \hat{m}, \quad 0 \leq \hat{s}.$$

The following algorithm computes a truncated finite-order spectral factor of degree $\hat{m}$ and $\hat{s}$ in the variable z and w, respectively.

7.4. Algorithm (Two-variable polynomial spectral factorisation). Input: *two-sided polynomial* $f(z, w)$. Output: *polynomial spectral factor* $g(z, w)$.

1. Define the $N_1 \times N_2$ matrix

$$\mathbf{f} = \begin{pmatrix} f_{0,0} & f_{0,1} & \cdots & f_{0,s} & 0 & \cdots & 0 & f_{0,s} & \cdots & f_{0,1} \\ f_{1,0} & f_{1,1} & \cdots & f_{1,s} & 0 & \cdots & 0 & f_{1,s} & \cdots & f_{1,1} \\ \vdots & \vdots & \vdots & \vdots & 0 & \cdots & 0 & \vdots & \vdots & \vdots \\ f_{m,0} & f_{m,1} & \cdots & f_{m,s} & 0 & \cdots & 0 & f_{m,s} & \cdots & f_{m,1} \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ f_{m,0} & f_{m,1} & \cdots & f_{m,s} & 0 & \cdots & 0 & f_{m,s} & \cdots & f_{m,1} \\ \vdots & \vdots & \vdots & \vdots & 0 & \cdots & 0 & \vdots & \vdots & \vdots \\ f_{1,0} & f_{1,1} & \cdots & f_{1,s} & 0 & \cdots & 0 & f_{1,s} & \cdots & f_{1,1} \end{pmatrix}$$

   with $N_1 \geq 2\,m+1$ and $N_2 \geq 2\,s+1$ both even.

2. Sample $f(z, z^{-1}, w)$ in $N_1 \times N_2$ points on the unit bicircle. Simply, perform 2-D FFT on $\mathbf{f}$ with $N_1 \times N_2$ interpolation points. Get a matrix $\mathbf{F}$.

3. Compute $\log(\mathbf{F})$ element by element to get a matrix $\mathbf{P}$.

4. Perform inverse 2-D FFT on $\mathbf{P}$ and get $\mathbf{p}$.

5. Let $\mathbf{b} = b_{i,j}$, $b_{1,j} = p_{1,j}/2$, $j = 1, \ldots, N_2$, $b_{i,j} = p_{i,j}$, $i = 2, \ldots, (N_1+1)/2$, $j = 1, \ldots, N_2$.

6. Perform 2-D FFT on $\mathbf{b}$ with $(N_1+1)/2 \times N_2$ interpolation points to get a matrix $\mathbf{B}$.

7. Compute $\exp(\mathbf{B})$ element by element and its inverse 2-D FFT to get a $(N_1 + 1)/2 \times N_2$ matrix whose upper-left $(\hat{m}+1) \times (\hat{s}+1)$ block has the structure

$$\begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,\hat{s}} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,\hat{s}} \\ \vdots & \vdots & \vdots & \vdots \\ g_{\hat{m},0} & g_{\hat{m},1} & \cdots & g_{\hat{m},\hat{s}} \end{pmatrix}.$$

$\square$

CHAPTER 8

Comparison to existing methods

In this chapter, we would like to compare our methods with others and show how our methods are effective or not. Like before, our aim is to control the spatially distributed system described by (3.2). In the first section, the system will be considered to be with lumped parameters and a method of basic course of control theory will be used to design a controller. In the second section, a method introduced in [10] and [8] will be used. In both experiments, we will watch the responses of the controlled system to initial conditions given in Fig. 3.5.

8.1. USE OF A TECHNIQUE FOR SYSTEMS WITH LUMPED PARAMETERS

The system (3.2) will be described by the state-space model of 1-D systems theory. LQG controller will then be designed. We again consider a rod with n actuators and sensors. The system has n states and can be described by equations

$$\begin{aligned} x_{k+1} &= A\,x_k + B\,u_k + w_k \\ y_k &= C\,x_k + D\,u_k + v_k \end{aligned}$$

with $x, u, y, w, v \in \mathbb{R}^n$, $A, B, C, D \in \mathbb{R}^{n\times n}$, where

$$A = \begin{pmatrix} 1-2\frac{T\kappa}{h^2} & \frac{T\kappa}{h^2} & 0 & \cdots & 0 \\ \frac{T\kappa}{h^2} & 1-2\frac{T\kappa}{h^2} & \frac{T\kappa}{h^2} & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \frac{T\kappa}{h^2} & 1-2\frac{T\kappa}{h^2} & \frac{T\kappa}{h^2} \\ 0 & \cdots & 0 & \frac{T\kappa}{h^2} & 1-2\frac{T\kappa}{h^2} \end{pmatrix},$$

$$B = T\,I, \qquad C = I, \qquad D = O,$$

where I and O denotes identity matrix and matrix of zeros, respectively, and the process noise w and measurement noise v are Gaussian white noises with covariance

$$\mathcal{E}\left\{w_k\, w_k^{\mathrm{T}}\right\} = \Phi_1, \qquad \mathcal{E}\left\{v_k\, v_k^{\mathrm{T}}\right\} = \Psi_1.$$

Now, we can approach the design of control and optimal estimator. Briefly speaking, the optimal controller in the sense of minimising the criteria

$$J = \sum_{k=0}^{\infty} x_k^{\mathrm{T}}\, \Psi\, x_k + u_k^{\mathrm{T}}\, u_k$$

will be designed, while the steady-state error

$$\mathcal{E}\left\{(x_k - \hat{x}_k)^{\mathrm{T}}(x_k - \hat{x}_k)\right\},$$

where $\hat{x}_k$ is the estimate of state, is minimised. For whole information, interested reader is referred to a relevant book, e. g. [43].

Let $n = 59$ and (3.10) and (3.11) holds. The LQG controller was designed for the noise covariance data $\Phi_1 = I$ and $\Psi_1 = I$ and for two different values of Ψ. The results are shown in the following figures. One can see that the responses are very similar to responses obtained by Alg. 6.14 in Sec. 6.3.

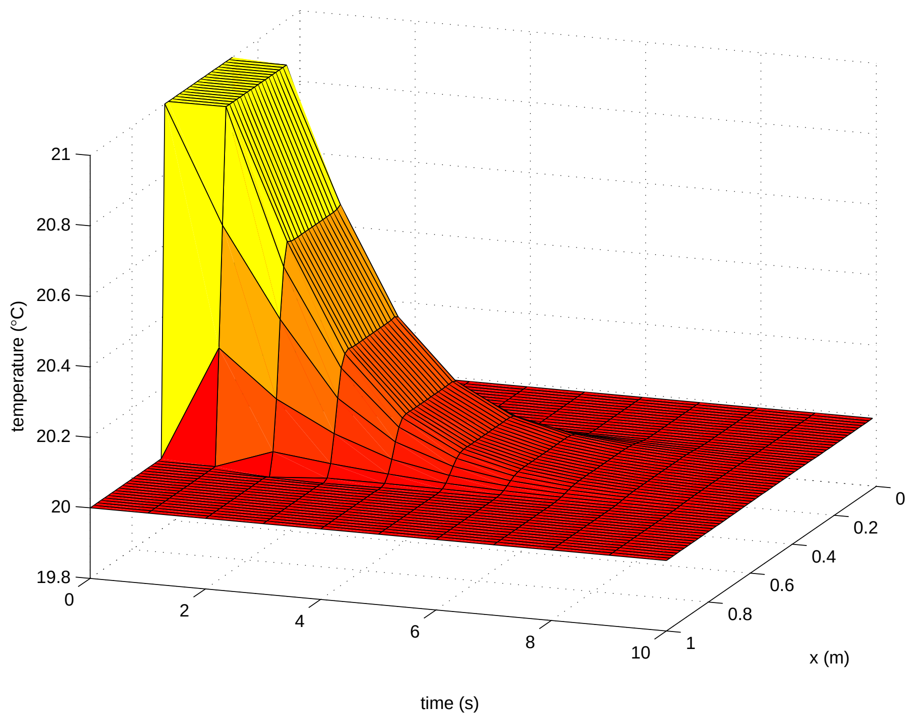

Fig. 8.1. Centralised LQG control, $\Psi = I$, output

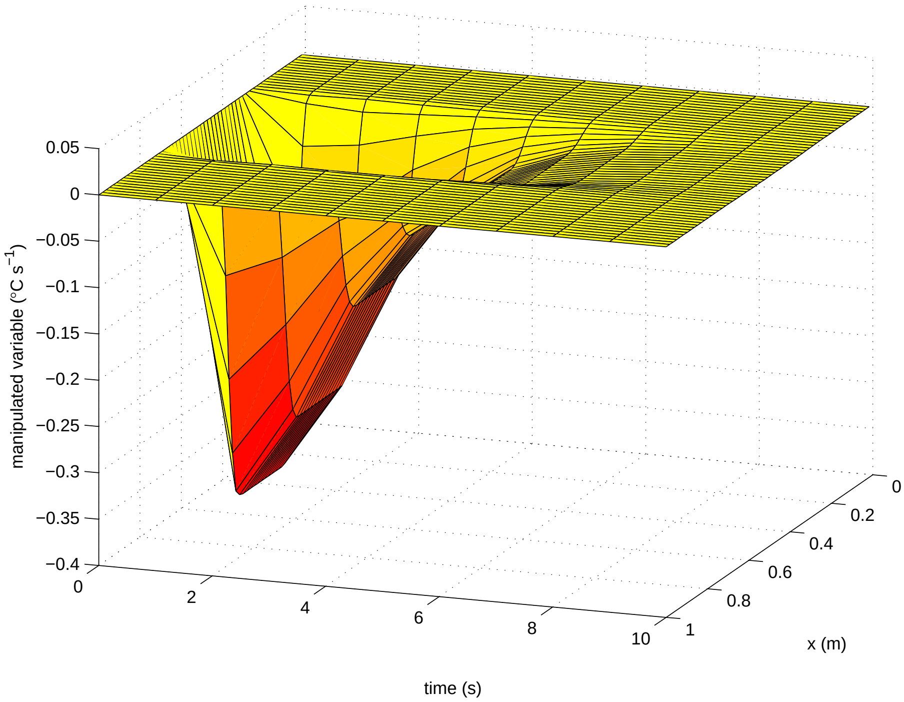

Fig. 8.2. Centralised LQG control, $\Psi = I$, manipulated variable

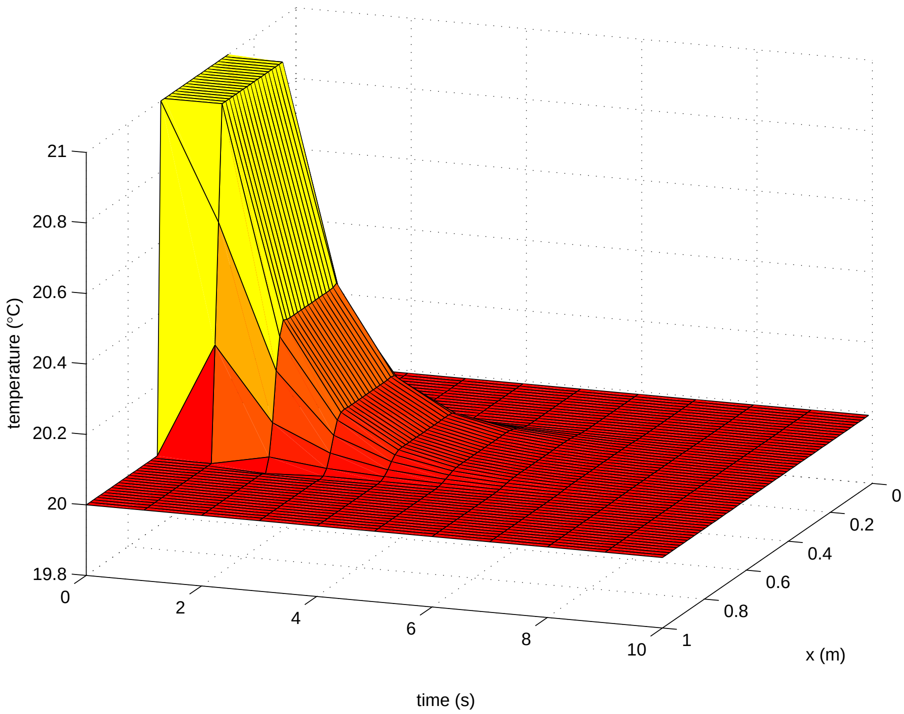

Fig. 8.3. Centralised LQG control,$\Psi = 100\, I$, output

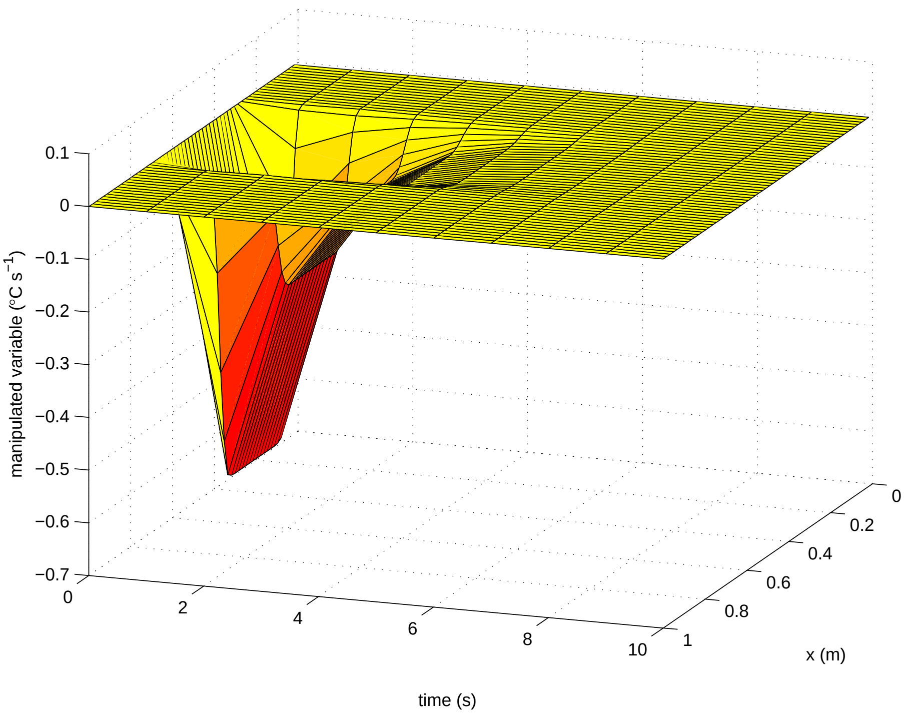

Fig. 8.4. Centralised LQG control, $\Psi = 100\,I$, manipulated variable

This control technique is based on a centralised controller. The control design requires manipulations with n by n matrices and can be very time-consuming. Relation between a number of heaters and sensors n and time needed to design LQG controller using Matlab command `lqg` is sketched in Fig. 8.5. One can see that for

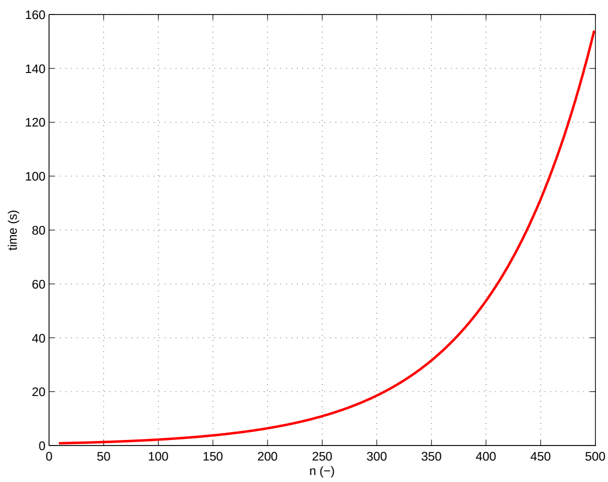

Fig. 8.5. Relation between number of nodes and centralised controller design time

high values of n this technique becomes impracticable. Another aspect is a time which calculation of the controller output takes during the control process. Operations with n by n matrices like adding, multiplying are needed. So, also from this point of view, the method is computationally hard. However, use of special algorithms for structured matrices can accelerate the computations. See e. g. [55].

The most time-consuming part of controller design using this technique is solving two Riccati equations, see [43] for details.

8.2. RICCATI EQUATION WITH POLYNOMIALS

In this section, we use a method described in [10] and elaborated e. g. in [8]. It consists in solving Riccati equation, however, unlike the previous section, with polynomial matrices. Before we start, we make the space discretisation of (3.2). Substituting

$$\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)_i = \frac{u(t)_{i-1} - 2\,u(t)_i + u(t)_{i+1}}{h^2}$$

into (3.2) we get

$$\left(\frac{\partial u(t)}{\partial t}\right)_i = \kappa \frac{u(t)_{i-1} - 2\,u(t)_i + u(t)_{i+1}}{h^2} + f(t)_i. \tag{8.1}$$

Performing the z-transform and manipulating, we can describe the system in the form of state-space equation as

$$\dot{u} = A\,u + B\,f, \tag{8.2}$$

with $A = \frac{\kappa}{h^2}\,(w - 2 + w^{-1})$, $w \in \mathbb{C}$, and $B = 1$.

As it is introduced in [10], the LQ controller can be designed by solving the algebraic Riccati equation

$$2\,A(w)\,P(w) - B(w)^2\,P(w)^2 + \Psi = 0, \tag{8.3}$$

where A, B, P generally depends on the complex variable w. The optimal control is then given by $f = -P\,u$, where P is the positive solution to (8.3).

Put $B = 1$. Let the positive solution to (8.3) be

$$P(w) = A(w) + \sqrt{A(w)^2 + \Psi}.$$

One can see that the optimal control is irrational. It cannot be used directly. A way how to deal with this problem is described in [48, 8].

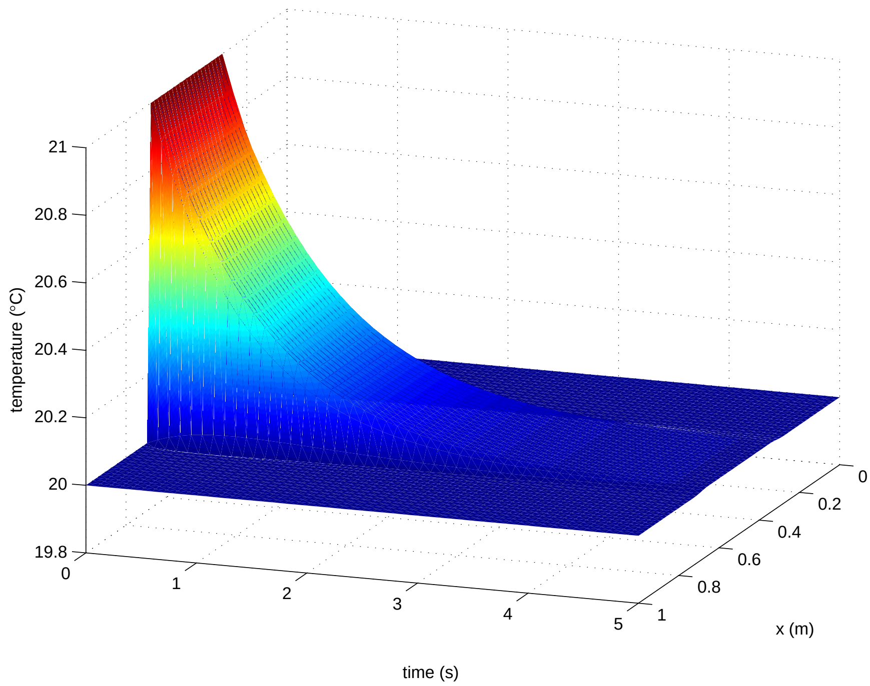

Fig. 8.6. Riccati equation with polynomials, output, $\Psi = 1$

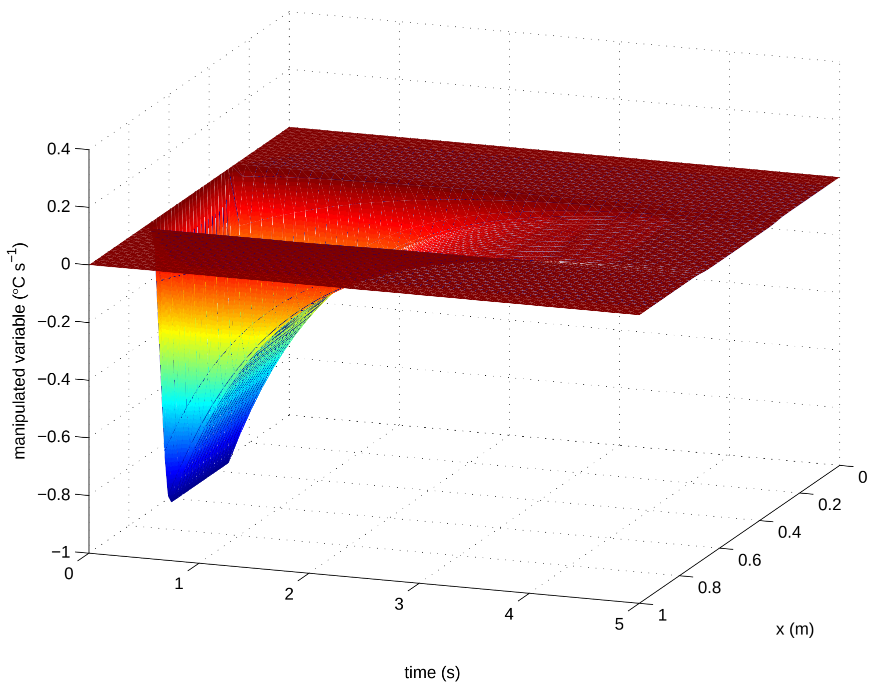

Fig. 8.7. Riccati equation with polynomials, manipulated variable, $\Psi = 1$

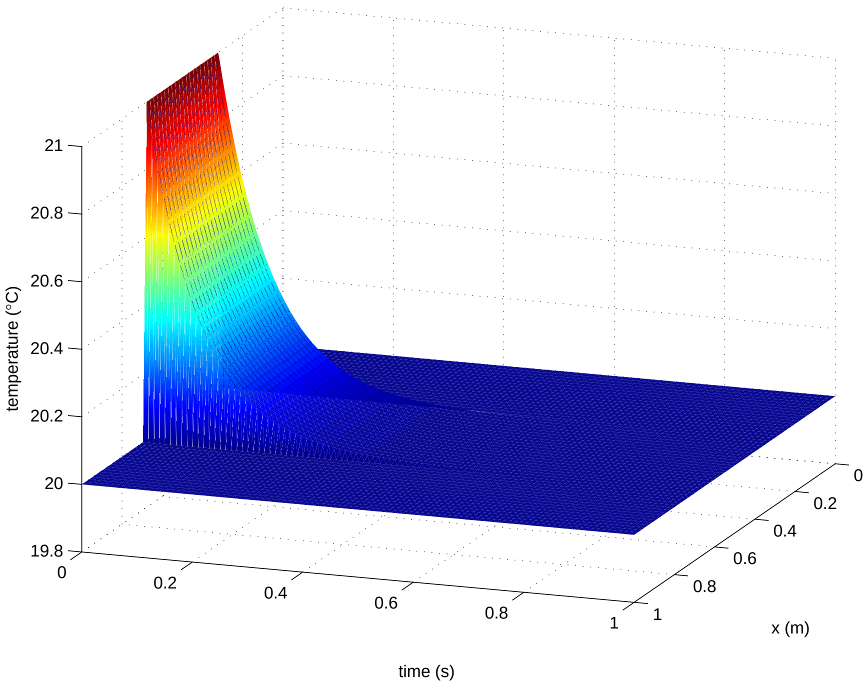

Fig. 8.8. Riccati equation with polynomials, output, $\Psi = 100$

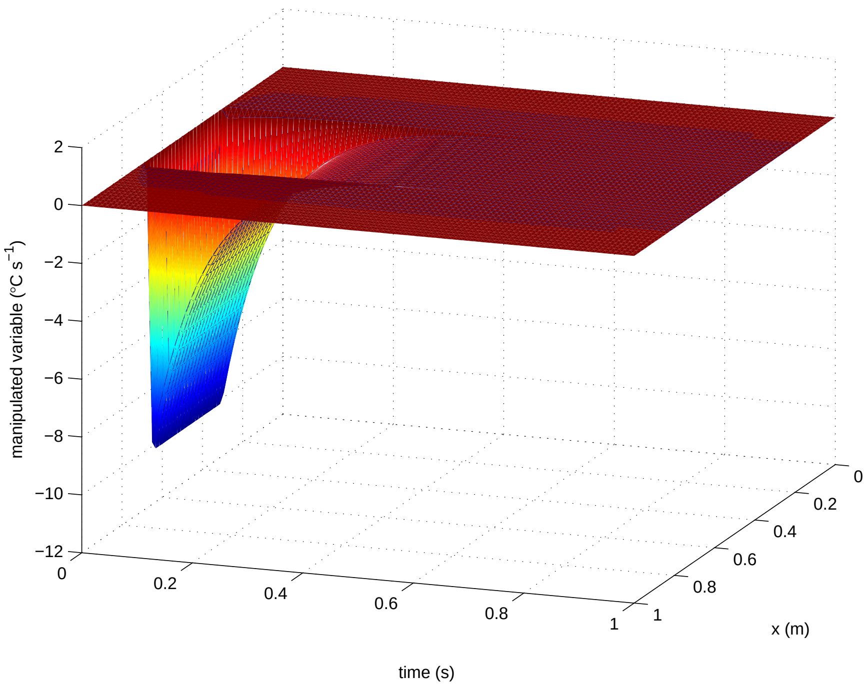

Fig. 8.9. Riccati equation with polynomials, manipulated variable, $\Psi = 100$

This approach gives the results in Figs. 8.6–8.9. Note that this technique leads to the control structure quite different from that obtained by our methods or method of the previous section. So, to compare them only watching the responses could be unfair. The following responses have shorter settling time, what is in accordance with higher power of manipulated variable. However, this difference occur since controller with no filter is considered in this control strategy.

The method presented in this section consists in solving Riccati equation with polynomials. Since the equation is scalar, the solution can be obtain quite easily. However, the solution is irrational and it must be approximated to be implemented in manner of distributed control.

8.3. CONCLUDING REMARKS

We compared the algorithm for design LQG control proposed in this thesis to two existing methods. The algorithms consists of the following operations.

Tab. 8.1. Essential operations for design of the LQG/LQ controller

centralised LQG controller	*Riccati equation with polynomials*	*our method (Alg. 6.14)*
2× Riccati equation with huge matrices	1× Riccati equation with polynomial matrices	2× spectral factorisation of a 2-D polynomial
	1× approximation of an irrational function	2× polynomial equation with 2-D polynomials

All methods are applicable to the same class of systems. The first method becomes impracticable for systems with high number of actuators and sensors. However, there are efficient algorithms for special types of matrices that arise in this method and using them can make the control design faster greatly.

The second method is an algorithm for LQ controller design based on solving Riccati equation with polynomials. Since a solution leads to irrational optimal control, one has to approximate it by a rational function to be able to implement the control using localised (distributed) controller.

The third method is Alg. 6.14. It consists of solving two spectral factorisations and two linear equations with 2-D polynomials.

CHAPTER 9

Conclusions

This thesis was dealing with modelling, analysis and control of spatially distributed systems. A method for discretisation of partial differential equation with constant coefficients was described in detail. Discrete-time and continuous-time models were derived. Corresponding transfer functions in the form of fraction of two-sided two-variable polynomial was obtained, assuming infinite spatial domain. In the thesis, the method was used for modelling a system described by a parabolic PDE, however, it works also for systems described by hyperbolic PDEs, as it was shown in [2, 12].

In Chap. 5, a positive polynomial approach to stability analysis of two-variable transfer functions was proposed. The problem of stability was formulated as the positivity of two-sided matrix polynomial. Using semidefinite programming formulation and the result of [72, 18] to solve this problem is believed to be new. Since the goal of this method is analysis of two-variable transfer functions, it is available for systems described by both parabolic and hyperbolic PDEs with the time variable and one spatial variable.

Stability analysis was extended to stabilisation in Secs. 6.1 and 6.2, where a stabilisable controllers were designed. Here, this method has restriction in the order of a plant. The order in the time variable is limited to be less than or equal to 1. The degree in the space variable can be arbitrary. However, many processes and phenomena in the nature are of order one. A function of the coefficients of characteristic polynomial can be brought in as a criterion and the controller can be optimised. It follows from the restriction that this method is useful for systems described by parabolic PDEs with one spatial variable. Using linearisation of product of two matrices, the method can also be used for systems of higher order, but the possibility to optimise a controller is lost. Then, it can also be used for systems described by hyperbolic PDEs with one spatial variable. These sections are completely new.

Sec. 6.3 deals with LQG control of spatially distributed systems. A theorem was proved. This was an extension of the 1-D case and the 2-D case of systems being of another class. A theorem and its proof for the spatially distributed systems are believed to be new. Simulations show that this approach can be used for systems described by a parabolic or hyperbolic PDE, see also [3].

The $\mathcal{H}_2$-optimal control and the dead-beat control technique were derived in Sec. 6.4 and 6.5, respectively. Methods presented here are extensions of techniques well-known in 1-D systems and can be used for systems described by a parabolic or hyperbolic PDE, see also [4].

Chap. 7 is scope of tools needed for control design from a computing point of view. In particular, linear equations with multivariate polynomials and multivariate polynomial spectral factorisation are handled. This part suites well-known principles to use them for spatially distributed systems.

Chap. 8 compares a method given in Chap. 6 to existing approaches. A plant was considered to be a system with lumped parameters and the LQG controller was designed using a method of theory of systems with lumped parameters. A method for control of spatially distributed systems based on works by Kamen [10] and Bamieh et al [8] was used.

9.1. ORIGINAL RESULTS AND CONTRIBUTIONS

The thesis brings some results that are believed to be original:

- Chap. 5. Multidimensional BIBO stability analysis.
- Sec. 6.1. Positive polynomial approach to stabilisation: discrete-time case.
- Sec. 6.2. Positive polynomial approach to stabilisation: continuous-time case.
- Sec. 6.3, Theorem 6.13 on LQG control and its proof for the spatially invariant systems.

The following papers were prepared within the thesis:

- The introduced method for modelling of spatially distributed systems has been described in conference paper [2] and in the journal *Mechanical systems and signal processing* [12]. Excluding self-citations, the journal paper [12] has 3 citations in the Web of Science databases, the paper [2] has 2 citations (not registered in the Web of Science databases).
- The methods for stability analysis and stabilisation using positive polynomial approach was submitted to the journal *Multidimensional systems and signal processing*.

- The algorithm for LQG control has been described in conference paper [3].
- The $\mathcal{H}_2$-optimal control has been presented in conference papers [4, 5]. Excluding self-citations, the paper [5] has 1 citation (not registered in the Web of Science databases).
- Some algorithms given in this thesis were implemented in Polmat [6, 1].

9.2. OUTLOOKS AND FUTURE WORK

The methods described in Chap. 5 and Secs. 6.1, 6.2 should be extended to more than one spatial variable. In such a case, the Schur-Cohn and Hermite-Fujiwara matrices become matrix polynomials in more than one variable and another approach has to be used to check if they are positive/sum-of-squares on the unit polycircle.

Removal of limitation to order in time of systems is also the subject of the future work. The obstacle here is a non-convexity of the set of parameters of stabilising controllers. It should be removed using non-convex optimisation or relaxation.

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