THESIS SUPERVISOR'S REPORT

I. IDENTIFICATION DATA

Thesis title:	**Nonsmooth Analysis and the Maximum Principle**
Author's name:	**Bc. Karolína Sehnalová**
Type of thesis :	master
Faculty/Institute:	Faculty of Electrical Engineering (FEE)
Department:	Department of Control Engineering
Thesis reviewer:	Doc. RNDr. Martin Bohata, Ph.D.
Reviewer's department:	Department of Mathematics

II. EVALUATION OF INDIVIDUAL CRITERIA

Assignment **challenging**

How demanding was the assigned project?

The assignment is devoted to a modern formulation of the Pontryagin maximum principle. It requires good knowledge of some parts of functional analysis and measure theory. Moreover, it is necessary to learn fundamentals of nonsmooth analysis. Let me note that the basic reference on nonsmooth analysis is a monograph aimed at graduate students and specialists. Therefore, I consider the assignment challenging.

Fulfilment of assignment **fulfilled**

How well does the thesis fulfil the assigned task? Have the primary goals been achieved? Which assigned tasks have been incompletely covered, and which parts of the thesis are overextended? Justify your answer.

All assigned tasks have been fulfilled.

Activity and independence when creating final thesis **A - excellent.**

Assess whether the student had a positive approach, whether the time limits were met, whether the conception was regularly consulted and whether the student was well prepared for the consultations. Assess the student's ability to work independently.

Ms Sehnalová had extremely positive approach. She worked with high level of independency and always attended our meetings well-prepared.

Technical level **A - excellent.**

Is the thesis technically sound? How well did the student employ expertise in his/her field of study? Does the student explain clearly what he/she has done?

Ms Sehnalová wrote the thesis mathematically sound. Abstract concepts of nonsmooth analysis are explained clearly and are accompanied by well-chosen examples. Discussion of considered optimal control problems contains right amount of pictures and graphs.

Formal level and language level, scope of thesis **A - excellent.**

Are formalisms and notations used properly? Is the thesis organized in a logical way? Is the thesis sufficiently extensive? Is the thesis well-presented? Is the language clear and understandable? Is the English satisfactory?

The thesis is very well written and organized. It consists of 94 pages (including the bibliography, two short appendices and the index of used symbols).

Selection of sources, citation correctness **A - excellent.**

Does the thesis make adequate reference to earlier work on the topic? Was the selection of sources adequate? Is the student's original work clearly distinguished from earlier work in the field? Do the bibliographic citations meet the standards?

Citations meet the usual mathematical standards. The references are well-chosen. They consist of important monographs as well as recent research articles.

Additional commentary and evaluation (optional)
Comment on the overall quality of the thesis, its novelty and its impact on the field, its strengths and weaknesses, the utility of the solution that is presented, the theoretical/formal level, the student's skillfulness, etc.

III. OVERALL EVALUATION, QUESTIONS FOR THE PRESENTATION AND DEFENSE OF THE THESIS, SUGGESTED GRADE

The thesis introduces the reader to a nice world of nonsmooth analysis with applications to optimal control theory. The main contribution of the work is given in the part on the optimal HIV treatment. In particular, using the hybrid maximum principle to a HIV treatment problem seems to be completely new. In my opinion, the work under review is excellent and fulfills all requirements for a Master's thesis.

The grade that I award for the thesis is **A - excellent.**

Date: **5.6.2024** Signature:

THESIS REVIEWER'S REPORT

I. IDENTIFICATION DATA

Thesis title:	**Nonsmooth Analysis and the Maximum Principle**
Author's name:	**Karolína Sehnalová**
Type of thesis :	master
Faculty/Institute:	Faculty of Electrical Engineering (FEE)
Department:	Department of Control Engineering
Thesis reviewer:	doc. Zdeněk Hurák, Ph.D.
Reviewer's department:	Department of Control Engineering FEE CTU in Prague

II. EVALUATION OF INDIVIDUAL CRITERIA

Assignment **challenging**

How demanding was the assigned project?

Challenging. The mathematical formalism of nonsmooth analysis is certainly well beyond the mathematical curriculum of a standard graduate engineering program.

Fulfilment of assignment **fulfilled**

How well does the thesis fulfil the assigned task? Have the primary goals been achieved? Which assigned tasks have been incompletely covered, and which parts of the thesis are overextended? Justify your answer.

The assigned goal was "to discuss", then "to focus on examples illustrating the applications". This goal has been attained.

Methodology **correct**

Comment on the correctness of the approach and/or the solution methods.

If I interpret the assigned "discussion" correctly as studying the necessary theory, identifying the necessary concepts and facts, and writing them down in the form of a tutorial containing also simple but well-developed examples, the methodology used in this thesis is correct.

Technical level **A - excellent.**

Is the thesis technically sound? How well did the student employ expertise in the field of his/her field of study? Does the student explain clearly what he/she has done?

In the first, overview part, the thesis is technically correct, which is mainly because the material is taken from established resources. The second part containing the examples also looks correct, at least as far as I have been able to judge.

A few minor remarks:

- In the *autonomous* (aka time-invariant) case, the Hamiltonian should not be written as a function of *t* as in the unnumbered equation following (3.7).
- The Forward-Backward Sweep Method described in the appendix A, which is taken from some literature about control of biological systems, resembles the method commonly known as just *gradient* or *steepest descent* method in classical optimal control texts such as Bryson and Ho 1975 (page 221), or Kirk 1970 (page 331). The sampled control trajectory is iterated over and the state and costate trajectories are solved using numerical solvers (forward and backward, respectively). The method is known to inherit poor convergence properties of the gradient method in the conventional nonlinear programming.
- It is claimed in the appendix B that "Runge-Kutta methods belong to the category of collocation method". Well, not all of them. It is the other way around: all collocation methods are (implicit) Runge-Kutta (IRK), but not all IRK are collocation methods.

Formal and language level, scope of thesis **A - excellent.**

Are formalisms and notations used properly? Is the thesis organized in a logical way? Is the thesis sufficiently extensive? Is the thesis well-presented? Is the language clear and understandable? Is the English satisfactory?

The thesis exhibits high writing culture (English, typesetting, …). But here are a few minor remarks:

- Including the current "organic" logo (the lion) of the CTU in the design of the front page is obviously a challenge.

- The notation $\langle x, y \rangle$ is used to denote two different concepts: the inner product of two vectors and the interval. I find this unnecessarily confusing. For the latter, the notation [x,y] could have been used instead.
- Page 28: I find the formulation of the optimal control problem (EP) rather unfortunate. What is meant here is that a minimum of J is search for and that J is defined like this and like that. But instead it reads that the minimum of J is equal to this and that... Similarly at several other places in the thesis.

Selection of sources, citation correctness **A - excellent.**

Does the thesis make adequate reference to earlier work on the topic? Was the selection of sources adequate? Is the student's original work clearly distinguished from earlier work in the field? Do the bibliographic citations meet the standards?

No major objections. I only doubt the usefulness of including URL for every paper in the bibliography, when DOI is included too. After all, the concept of DOI was invented exactly to avoid the need to rely on URL, which can change without notice.

And I can also recommend some additional resources for nonsmooth optimization and proximal "stuff", such as

- Beck, Amir. First-Order Methods in Optimization. MOS-SIAM Series on Optimization. Society for Industrial and Applied Mathematics, 2017.
- Parikh, Neal, and Stephen Boyd. "Proximal Algorithms." Found. Trends Optim. 1, no. 3 (January 2014): 127–239. https://doi.org/10.1561/2400000003.
- Nesterov, Yurii. Lectures on Convex Optimization. 2nd ed. Springer Optimization and Its Applications. Cham: Springer, 2018.

But indeed, all the fundamental nonsmooth analysis stuff that was needed can be found in Clarke's materials. The above references are essentially meant just as another inspiration…

Additional commentary and evaluation (optional)

Comment on the overall quality of the thesis, its novelty and its impact on the field, its strengths and weaknesses, the utility of the solution that is presented, the theoretical/formal level, the student's skillfulness, etc.

On the one hand I very much appreciate that the student decided to step into this advanced area of mathematical control theory and used the diploma thesis/project for this. On the other hand, it should have been even more directly communicated that indeed, no new concepts have been invented, no new theorems have been proven. Well, at least if I have understood it all correctly…

The thing is that at a few places the thesis may give a false impression that the results are claimed as new. For example, the proof of the Theorem 2.3 on page 7 must have already been published elsewhere. And similarly, several other propositions and theorems for which no references are listed. Are all these really claimed as new? If yes, then I apologize for my ignorance. This terminological confusion is also reflected in the Conclusions, where it is stated that "In this thesis, nonsmooth analysis was introduced in order to formulate the extended maximum principle". But strictly speaking, this is a bit misleading, isn't it? Nonsmooth analysis has been *introduced* elsewhere and by someone else, right? In this thesis, all this formalism was, say, overviewed, summarized, and finally demonstrated. Even more confusing is the Section 3.2 on the Extended PMP. On page 27 the author writes "The extended maximum principle is the most general maximum principle we provide in this chapter", and then on the next page she writes "Our assumptions are sterner compared to what Clarke assumes in [Cla13]." This suggests that the author really claims some novel (maybe simplified?) proof for stricter assumptions than what can be found in the literature. But the on the yet another page she finally writes "Now we formulate the extended principle, proved in [Cla13, p. 514]", which seems to reinforce my initial interpretation of this whole theoretical section.

Let me make it straight that even when formulated like that, I regard such contribution more than enough for a master thesis within an engineering study program.

CTU
CZECH TECHNICAL UNIVERSITY IN PRAGUE

THESIS REVIEWER'S REPORT

III. OVERALL EVALUATION, QUESTIONS FOR THE PRESENTATION AND DEFENSE OF THE THESIS, SUGGESTED GRADE

Summarize your opinion on the thesis and explain your final grading. Pose questions that should be answered during the presentation and defense of the student's work.

Within this master thesis, the student demonstrates that she has mastered the very advanced mathematical formalism of nonsmooth analysis that is needed to formulate, understand and use nonsmooth version(s) of Pontryagin's Maximum Principle. She does it by providing an overview of the necessary background material first, then stating the extended result(s) published in the literature, and finally applying these results to two examples. I regard the thesis a success.

If time permits during the defense, the student should answer (any of) the following **questions**:

1. Could you perhaps recapitulate the attributes of the *boat example* that called for the extended PMP because they could not be served just with the standard PMP?
2. How would you now, after all this studying, identify the pros and cons of the indirect techniques (PMP-based) with respect to the direct methods (relying heavily on discretization followed by nonlinear programming)?
3. You direct the reader at one point in the thesis to some literature for *more* information about *reachability*. But the fact is that you offer no discussion of this concept at all. Could you at least sketch how we can determine if the target set for the optimal control problem is reachable in the first place?

The grade that I award for the thesis is **A - excellent.**

Date: **16.6.2024** Signature: