Czech Technical University in Prague
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Habilitation thesis

Analysis and Synthesis of Systems with Fuzzy Parametric Uncertainty

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To Martina, an amazing human being

Contents

Chapter 1

Introduction

Control systems are usually designed such that they perform optimally in some sense in an operating point. However, they have to maintain certain essential properties even under presence of large perturbations. Uncertainty is inherent to any model since it represents only an approximation of a real system.

If a mathematical model of such system exists and the value of some of its parameter is fixed but not known exactly the corresponding problems are typically solved in the framework of robust analysis and synthesis of systems with parametric uncertainty ([1, 2]). The parameters of a linear system are supposed to lie within the intervals with nominal values usually being in the middles. Nevertheless, many times an estimation of the value of an uncertain parameter is given by an expert who usually uses a linguistic description (e.g. "more or less 5"). In this case, it is natural to represent an uncertain parameter q_i by a *fuzzy number*, $\tilde{q}_i$, with membership function $\alpha = \mu_{\tilde{q}_i}(q_i) \in [0, 1]$ that is convex, normal with bounded support. The membership value α can be interpreted as the confidence degree in that the value of the parameter equals its nominal value. The uncertainty interval is supposed to be parameterized by a confidence level. When a confidence level α is specified then each parameter interval is determined by the α-cut $[\tilde{q}_i]_\alpha$. If $\alpha = 1$ (the maximum confidence level – the system works in normal or *most-cases* operating conditions) the parameter q_i equals to the core of $\tilde{q}_i$, $q_i = \mathrm{core}(\tilde{q}_i)$. If $\alpha = 0$ (the minimum confidence level corresponding to *worst-case* operating conditions) the parameter q_i can take any value within the support of $\tilde{q}_i$, $q_i \in \mathrm{supp}(\tilde{q}_i)$. The systems whose parameters are described by fuzzy numbers are referred to as the systems with fuzzy parametric uncertainty.

Generally, a continuous system with fuzzy parametric uncertainty with single input and single output is described by the differential equation

$$f(y(t),\ldots,y^{(n)}(t),u(t),\ldots,u^{(m)}(t),\tilde{\mathbf{q}})=0 \tag{1.1}$$

where $\tilde{\mathbf{q}}=(\tilde{q}_1,\ldots,\tilde{q}_r)$, $\tilde{q}_i\in\tilde{P}(\Re)$, $i=1,\ldots,r$, is a vector of uncertain parameters described by fuzzy numbers and $\tilde{P}(\Re)$ denotes the set of all possible fuzzy sets with real universe of discourse.

The linear system with fuzzy parametric uncertainty is described by the linear differential equation

$$a_n(\tilde{\mathbf{q}})y^{(n)}(t)+a_{n-1}(\tilde{\mathbf{q}})y^{(n-1)}(t)+\cdots+a_0(\tilde{\mathbf{q}})y(t)=b_m(\tilde{\mathbf{q}})u^{(m)}(t)+\cdots+b_0(\tilde{\mathbf{q}})u(t) \tag{1.2}$$

where $a_i(\cdot)$, $b_j(\cdot)$ are fuzzy functions.

The transfer function of the system (1.2) can be defined as

$$\tilde{G}(s,\tilde{\mathbf{q}})=\frac{Y(s)}{U(s)}=\frac{b_m(\tilde{\mathbf{q}})s^m+b_{m-1}(\tilde{\mathbf{q}})s^{m-1}+\ldots+b_0(\tilde{\mathbf{q}})}{a_n(\tilde{\mathbf{q}})s^n+a_{n-1}(\tilde{\mathbf{q}})s^{n-1}+\ldots+a_0(\tilde{\mathbf{q}})}. \tag{1.3}$$

If a degree of confidence in the coefficients is given, the system (1.3) can be represented as an interval system described by the family of transfer functions

$$[\tilde{G}(s,\tilde{\mathbf{q}})]_\alpha=\frac{b_m([\tilde{\mathbf{q}}]_\alpha)s^m+b_{m-1}([\tilde{\mathbf{q}}]_\alpha)s^{m-1}+\ldots+b_0([\tilde{\mathbf{q}}]_\alpha)}{a_n([\tilde{\mathbf{q}}]_\alpha)s^n+a_{n-1}([\tilde{\mathbf{q}}]_\alpha)s^{n-1}+\ldots+a_0([\tilde{\mathbf{q}}]_\alpha)} \tag{1.4}$$

where $[\tilde{\mathbf{q}}]_\alpha=([\tilde{q}_1]_\alpha,\ldots,[\tilde{q}_r]_\alpha)$, $[\tilde{q}_i]_\alpha=[q_{i\alpha}^-,\,q_{i\alpha}^+]$, $i=1,\ldots,r$ is a vector of interval elements corresponding to the α-cuts of the parameters $\tilde{q}_i$.

Characteristic polynomial of the system (1.4) can be written as

$$p(s,\tilde{\mathbf{q}})=a_0(\tilde{\mathbf{q}})+a_1(\tilde{\mathbf{q}})s+\ldots+a_n(\tilde{\mathbf{q}})s^n. \tag{1.5}$$

If the polynomial (1.5) has independent uncertainty structure, i.e. each parameter $\tilde{q}_i$ appears in only one coefficient $a_j(\tilde{\mathbf{q}})$, then it can be represented by the α-cuts

$$[p(s,\tilde{\mathbf{q}})]_\alpha=\sum_{i=0}^{n}[a_{i\alpha}^-,a_{i\alpha}^+]s^i \tag{1.6}$$

where

$$a_{i\alpha}^-=\min_{\mathbf{q}\in[\tilde{\mathbf{q}}]_\alpha}a_i(\mathbf{q}),\quad a_{i\alpha}^+=\max_{\mathbf{q}\in[\tilde{\mathbf{q}}]_\alpha}a_i(\mathbf{q}). \tag{1.7}$$

However, the coefficients of the transfer function may not be necessarily independent as in (1.6). The parameters can enter into the coefficients of the transfer function and characteristic polynomial in linear, multilinear, polynomic or even more complicated manner. Therefore a mathematical framework for computation with fuzzy numbers is needed. Such a framework was proposed by Bondia and Picó in [3] and [4]. They adopted the concept of fuzzy functions, see [5], and fuzzy arithmetic [6, 7].

Modelling of uncertain parameters by fuzzy numbers has some significant advantages over the classical robust control approach using intervals that makes it more suitable in many practical applications. Firstly, the classical approach assumes that the uncertainty variation remains the same independently on the distance from a nominal point. However, in many practical situations the uncertainty varies depending on the operation conditions. The use of weighted norms partially overcomes this problem but the weights of each coefficient must be chosen a priori by heuristics, leading to a trial-and-error procedure when not satisfactory results are obtained. Secondly, the classical approach considers only the case when the parameter values corresponding to an operating point lie in the middle of their admissible intervals which is very often not true (see example 3.1).

Recently, various problems relating to linear systems with fuzzy parametric uncertainty have been solved. In [8] controller synthesis for such systems under fuzzy pole placement specifications is suggested. Several approaches of simulation of fuzzy discrete-time systems with uncertain initial state are proposed in [9, 10]. A practical application of systems with parametric uncertainty characterized by fuzzy numbers is described in [11]. One of the fundamental problems associated to those systems is to determine what minimum confidence level $\alpha_{\min}$ guarantees stability of the system under the assumption that the nominal system (i.e. for $\alpha = 1$) is Hurwitz stable. A different definition for a measure of fuzzy system stability based on the degree of belief that a system is stable was proposed in [12].

The thesis addresses some problems of analysis and synthesis of linear systems with fuzzy parametric uncertainty. In chapter 2 and 3 the in robust control frequently encountered problem of finding stability margin of a characteristic polynomial is solved. Chapter 2 deals with that problem for systems with independent uncertainty structure. The task consists in determination of minimum confidence level $\alpha_{\min}$ such that the characteristic polynomial (1.7) is Hurwitz stable for $\alpha > \alpha_{\min}$ and unstable for $\alpha \leq \alpha_{\min}$. The first part supposes that each coefficient is described by one fuzzy number whereas the second part is concerned with a multivariate membership function characterizing all the coefficients. The chapter extends and improves the results achieved in [13] and [14]. Chapter 3 is concerned with the same problem for characteristic polynomials with linear parameter dependency, i.e. the case when all the coefficients of the characteristic polynomial are linear affine functions of the parameters represented by fuzzy numbers. Linear parameter dependency makes it possible to consider more realistic cases including feedback control of a linear system with fuzzy parametric uncertainty by a fixed controller. The results are derived for parameters described by arbitrary membership functions followed by their simplification for nonsymmetric triangular membership functions. Main idea of the methodology was presented in [15].

Chapters 4 and 5 address the problem of PID controllers design for linear systems with fuzzy parametric uncertainty. The typical task of robust control approach to systems with parametric uncertainty is to find a controller such that some closed loop requirements are met for the whole set of admissible plants. The disadvantage of the classical approach is that uncertainty of the plant is considered the same regardless of the operating conditions. Therefore the *worst-case* uncertainty occurring very rarely only in unusual conditions has to be taken into account with the same importance as *most-cases* uncertainty caused by the influence of common factors. However, in practical applications the closed loop specifications (e.g. maximum step response overshoot, maximum settling time) are usually stronger for the system that operates in typical conditions than those for the system with its parameters lying far from the normal operating point. Consequently the controller designed such that it satisfies the worst-case specifications may not lead to satisfactory performance for the most-cases model.

Since fuzzy numbers can be interpreted as possibility distributions the confidence level can be also seen as a possibility value associated to every model. The models can arise from identification experiments assigning possibility distributions for the system parameters. Therefore, using fuzzy numbers to represent both uncertain parameters and closed loop specifications offers an elegant tool how to degrade behaviour of the feedback system towards the uncertainty that occurs very rarely.

Chapter 4 deals with the problem of PI and PD controller design for linear systems with fuzzy parametric uncertainty with interval phase and gain margin specifications. A shorter version of the chapter will appear in [16]. Chapter 5 addresses the problem of PID controller design with sensitivity and complementary sensitivity peak specifications that proved themselves to be more reliable indicator of closed loop behaviour than phase or gain margins.

Chapter 2

Linear systems with fuzzy parametric uncertainty

2.1 Introduction

When dealing with real systems it is not possible to obtain an accurate model of a system, some uncertainty has to be always considered. If the structure of a system is supposed to be given but the parameters are not known precisely we speak about parametric uncertainty. In engineering practice it is of fundamental importance that the systems preserve stable behaviour for a whole admissible parameter variations. In this view it could be also appropriate to know, if a system is stable for some nominal values of its parameters, within what boundary the stability remains preserved. Such a problem is called *stability margin* determination.

Since the celebrated Kharitonov theorem [17] was published big attention is devoted to solving both problems – checking stability of the uncertain system and determining its stability margin. Kharitonov theorem provides very efficient tool for stability analysis of interval systems, i.e. linear systems whose coefficients are supposed to lie in the prescribed mutually independent intervals. To check stability of a system with linear parameter dependency the Edge theorem [18] provides a feasible solution. More complicated coefficient structures such as multilinear or polynomic dependency on an interval vector parameter are also considered, however the corresponding algorithms are rather complicated.

When linear systems with fuzzy parametric uncertainty are considered stability margin is equivalent to minimum confidence level needed to preserve stability.

2.2 Interval fuzzy linear systems

Let us consider a linear system with its parameters described by fuzzy numbers entering the coefficients of the characteristic polynomial independently. Such polynomial can be written as

$$\tilde{p}(s) = \tilde{q}_0 + \tilde{q}_1 s + \ldots + \tilde{q}_n s^n \tag{2.1}$$

where the coefficients $\tilde{q}_k,\ k = 0, \ldots, n$ are described by fuzzy sets with membership functions $\mu_{\tilde{q}_k}(q_k)$.

To be able to use techniques known from robust control theory it is convenient to represent the varying intervals expressed by the α-cuts by parameterization of varying endpoints of these intervals. If convex membership functions are used it is always possible to write $[\tilde{q}_k]_\alpha = [q_k^-(\alpha), q_k^+(\alpha)] \stackrel{\text{def}}{=} q_k(\alpha)$ where α is the confidence level and $q_k^-(\cdot)$ and $q_k^+(\cdot)$ is strictly increasing and strictly decreasing function, respectively. If $q_k^0 = \text{core}(\tilde{q}_k)$ and $[q_k^-, q_k^+] = \text{supp}(\tilde{q}_k)$ then

$$\begin{aligned} q_k^-(\alpha) &= \mu_{\tilde{q}_k}^{-1}(\alpha) \quad \text{for} \quad q_k \le q_k^0, \\ q_k^+(\alpha) &= \mu_{\tilde{q}_k}^{-1}(\alpha) \quad \text{for} \quad q_k \ge q_k^0. \end{aligned}$$

The functions $q_k^-(\alpha)$ and $q_k^+(\alpha)$ satisfy $q_k^-(0) = q_k^-$, $q_k^+(0) = q_k^+$ and $q_k^-(1) = q_k^+(1) = q_k^0$.

2.3 Problem statement

The α-cut representation of polynomial (2.1) is defined as an interval polynomial

$$[\tilde{p}(s)]_\alpha = p(s, \alpha) = \sum_{k=0}^{n} q_k(\alpha) s^k \tag{2.2}$$

where $q_k(\alpha) = [q_k^-(\alpha), q_k^+(\alpha)]$.

Let us note that for any $\alpha > 0$ the corresponding coefficient space of polynomial (2.2) is a hyperrectangle.

Let us suppose that the nominal (1-cut) polynomial $[\tilde{p}(s)]_{\alpha=1} = p(s,1) = \sum_{i=0}^{n} q_i^0 s^i$ is stable. The task is to find stability margin of the polynomial (2.1), i.e. confidence level $\alpha_{\min} \in [0,1]$ such that interval polynomial (2.2) is stable for $\alpha > \alpha_{\min}$ and unstable for $\alpha \leq \alpha_{\min}$.

Similar problems have been solved using a binary search in [12] or using Argoun stability test [19] in [4] or with the help of Kharitonov theorem [17]. However they are graphical in nature or require time-consuming numerical algorithms even for a simple shape of membership functions. In this chapter we introduce more elegant solution based on generalization of Tsypkin-Polyak locus [20].

2.4 Determination of stability margin

First let us remind well-known result of robust control theory.

2.4.1 Zero exclusion principle

Let us consider a family of polynomials

$$\begin{aligned} p(s,A) &= a_0 + a_1 s + \cdots + a_n s^n, \\ \mathbf{a} &= [a_0, \ldots, a_n]^{\mathrm{T}}, \mathbf{a} \in A \end{aligned} \tag{2.3}$$

where $A \subset \Re^{n+1}$ is a compact set.

Theorem 2.1 (Zero exclusion principle): *The family of polynomials $p(s,A)$ (2.3) of invariant degree is D-stable if and only if*
a) there exists a D-stable polynomial $p(s,\mathbf{a}^*)$, $\mathbf{a}^* \in A$,
b) $0 \notin p(s^*,A) \forall s^* \in \partial D$

where ∂D stands for boundary of D. In the case of Hurwitz stability ∂D corresponds to the imaginary axis (semiaxis) of the complex plane. The set $p(s^*,A), s^* \in \partial D$ is called the value set.

Considering $p_1(\omega) = h(\omega) + jg(\omega)$ instead of $p(j\omega) = h(\omega) + j\omega g(\omega)$ where $h(s)$ and $sg(s)$ are the even and odd parts of the polynomial $p(s)$ respectively a variation of Zero exclusion principle can be stated.

Theorem 2.2: *The family of polynomials* $p(s, A)$ *(2.3) containing a Hurwitz polynomial* $p(s, \mathbf{a}^*)$, $\mathbf{a}^* \in A$ *is Hurwitz stable if and only if*
a) the coefficient a_n *does not include 0,*
b) for $\omega = 0$ *the value set* $p_1(\omega, A)$ *does not include points on the imaginary axis,*
c) $0 \notin p_1(\omega, A) \forall \omega \geq 0$.

Remark 2.1 From the monotonic phase increase property for Hurwitz polynomials follows that the frequency plot of $p(s, \mathbf{a}^*)$ in the complex plane goes through n quadrants in the counterclockwise direction.
Remark 2.2 The part b) of the theorem 2.2 is equivalent to the condition that the coefficient a_0 does not include 0 because $h(0) = 0$ is equivalent to $a_0 = 0$.

Since dividing of the even and odd parts of a polynomial by some positive functions cannot affect zero exclusion or inclusion in value set we can replace $p_1(\omega)$ by $p_2(\omega) = h(\omega)/S(\omega) + jg((\omega)/T(\omega))$ where $S(\omega)$ and $T(\omega)$ are positive functions of $\omega \geq 0$. Moreover, if $\lim_{\omega\to\infty} h(\omega)/S(\omega)$ and $\lim_{\omega\to\infty} g(\omega)/T(\omega)$ are finite we can replace the condition c) of the theorem 2.2 by the condition c) of the following theorem.

Theorem 2.3: *The family of polynomials* $p(s, A)$ *(2.3) containing a Hurwitz polynomial* $p(s, \mathbf{a}^*)$, $\mathbf{a}^* \in A$ *is Hurwitz stable if and only if*
a) the coefficient a_n *does not include 0,*
b) the coefficient a_0 *does not include 0,*
c) $0 \notin p_2(\omega, A) \forall \omega \geq 0$.

2.4.2 Generalized Tsypkin-Polyak plot

Let us define strictly decreasing functions

$$\begin{aligned} r_k^-(\alpha) &= q_k^0 - q_k^-(\alpha), \\ r_k^+(\alpha) &= q_k^+(\alpha) - q_k^0. \end{aligned} \tag{2.4}$$

Consider an uncertain polynomial

$$\begin{aligned} p(s,Q) &= q_0 + q_1 s + \cdots + q_n s^n, \mathbf{q} = [q_0, \ldots, q_n]^{\mathrm{T}}, \\ Q &= \left\{ \mathbf{q} : \left[\sum_{k=0}^{n} \left| \frac{q_k - q_k^0}{r_k(\alpha)} \right|^p \right]^{\frac{1}{p}} \le 1 \right\} \end{aligned} \tag{2.5}$$

where

$$r_k(\alpha) = \begin{cases} r_k^-(\alpha), & \text{if } q_k < q_k^0 \\ r_k^+(\alpha), & \text{if } q_k \ge q_k^0 \end{cases}$$

and p is an arbitrary positive integer.

It is worth noting that for $p = \infty$ the uncertain polynomial (2.5) equals to the α-cut polynomial (2.2).

Let us again decompose a member of family of polynomials (2.5) into its even and odd part. For $s = \mathrm{j}\omega$ we can write

$$p(\mathrm{j}\omega, \mathbf{q}) = h(\omega, \mathbf{q}) + \mathrm{j}\omega g(\omega, \mathbf{q}), \mathbf{q} \in Q. \tag{2.6}$$

The nominal polynomial $p_0(s)$ evaluated at $s = \mathrm{j}\omega$ then can be written as

$$p_0(\mathrm{j}\omega) = p(\mathrm{j}\omega, \mathbf{q}^0) = h_0(\omega) + \mathrm{j}\omega g_0(\omega) \tag{2.7}$$

where

$$\begin{aligned} h_0(\omega) &= q_0^0 - q_2^0\omega^2 + q_4^0\omega^4 - \cdots, \\ g_0(\omega) &= q_1^0 - q_3^0\omega^2 + q_5^0\omega^4 - \cdots. \end{aligned} \tag{2.8}$$

Denote

$$\begin{aligned} S_p(\omega,\alpha) &= 0.5\left(S_p^-(\omega,\alpha) + S_p^+(\omega,\alpha)\right) + 0.5\left(S_p^-(\omega,\alpha) - S_p^+(\omega,\alpha)\right)\operatorname{sgn} \mathrm{h}_0(\omega), \\ S_p^+(\omega,\alpha) &= \left(\sum_{k=0}^{n/4} \left(r_{4k}^+(\alpha)\omega^{4k}\right)^q + \sum_{k=0}^{(n-2)/4} \left(r_{4k+2}^-(\alpha)\omega^{4k+2}\right)^q \right)^{\frac{1}{q}}, \\ S_p^-(\omega,\alpha) &= \left(\sum_{k=0}^{n/4} \left(r_{4k}^-(\alpha)\omega^{4k}\right)^q + \sum_{k=0}^{(n-2)/4} \left(r_{4k+2}^+(\alpha)\omega^{4k+2}\right)^q \right)^{\frac{1}{q}}, \\ T_p(\omega,\alpha) &= 0.5\left(T_p^-(\omega,\alpha) + T_p^+(\omega,\alpha)\right) + 0.5\left(T_p^-(\omega,\alpha) - T_p^+(\omega,\alpha)\right)\operatorname{sgn} \mathrm{g}_0(\omega), \\ T_p^+(\omega,\alpha) &= \left(\sum_{k=0}^{(n-1)/4} \left(r_{4k+1}^+(\alpha)\omega^{4k}\right)^q + \sum_{k=0}^{(n-3)/4} \left(r_{4k+3}^-(\alpha)\omega^{4k+2}\right)^q \right)^{\frac{1}{q}}, \\ T_p^-(\omega,\alpha) &= \left(\sum_{k=0}^{(n-1)/4} \left(r_{4k+1}^-(\alpha)\omega^{4k}\right)^q + \sum_{k=0}^{(n-3)/4} \left(r_{4k+3}^+(\alpha)\omega^{4k+2}\right)^q \right)^{\frac{1}{q}} \end{aligned} \tag{2.9}$$

where q is the index conjugate to p:

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{2.10}$$

Without loss of generality suppose $q_n^0 > 0$. Then the key theorem can be stated.

Theorem 2.4: *Denote by α_∞ and α_0 the solutions of*
a) $q_n^0 = r_n^-(\alpha)$,
b) $q_0^0 = r_0^-(\alpha)$,
respectively, and by α_ω the solutions of
c) $\max\left\{\frac{h_0(\omega)}{S_\infty(\omega,\alpha)}, \frac{g_0(\omega)}{T_\infty(\omega,\alpha)}\right\} = 1$ *with respect to α for each $\omega > 0$,*
on the interval $\alpha \in [0,1]$. Assign zero in the case that a solution does not exist.

Then

$$\alpha_{\min} = \max\{\alpha_\infty, \alpha_0, \underline{\alpha}\} \tag{2.11}$$

where $\underline{\alpha} = \sup_{\omega>0} \alpha_\omega$.

Remark 2.3 For $q_n^0 < 0$ the $r_n^-(\cdot)$ and $r_0^-(\cdot)$ are replaced by $r_n^+(\cdot)$ and $r_0^+(\cdot)$, respectively.

Proof: Since both $S_p(\omega,\alpha)$ and $T_p(\omega,\alpha)$ are positive for $\omega \geq 0$ and the values $\lim_{\omega\to\infty} h(\omega,\mathbf{q})/S_p(\omega)$ and $\lim_{\omega\to\infty} g(\omega,\mathbf{q})/T_p(\omega)$ are finite for all $p(s,\mathbf{q}), \mathbf{q} \in Q$ defined by (2.5) theorem 2.3 can be applied. Since all $r_n^-(\alpha)$, $r_0^-(\alpha)$ and $S_p(\omega,\alpha)$, $T_p(\omega,\alpha)$ for a fixed ω are decreasing functions of α the values α_∞, α_0 and α_ω for each $\omega > 0$ are unique. The justification of α_∞ and α_0 is obvious from the conditions a) and b). To show the role of $\underline{\alpha}$ we will explore when zero is excluded from the value set (the condition c)).

Denote by $\Delta q_k = q_k - q_k^0$ and by $\delta_k(\alpha) = \Delta q_k / r_k^+(\alpha)$ for $\Delta q_k \geq 0$, $\delta_k(\alpha) = \Delta q_k / r_k^-(\alpha)$ for $\Delta q_k < 0$. The deviations of even and odd parts of a polynomial then can be expressed as

$$\begin{aligned} \Delta h(\omega) &= h(\omega,\mathbf{q}) - h_0(\omega) = \sum_{k=0}^{n/2} (-1)^k \Delta q_{2k} \omega^{2k}, \\ \Delta g(\omega) &= g(\omega,\mathbf{q}) - g_0(\omega) = \sum_{k=0}^{(n-1)/2} (-1)^k \Delta q_{2k+1} \omega^{2k}, \end{aligned}$$

respectively.

Let us discuss four different cases according to the signs of $\Delta h(\omega)$ and $\Delta g(\omega)$.

1. $\Delta h(\omega) \geq 0, \Delta g(\omega) \geq 0$:

For $\Delta h(\omega) \geq 0$ we can write

$$\Delta h(\omega) \;\leq\; \sum_{k=0}^{n/4} \delta_{4k}(\alpha) r_{4k}^{+}(\alpha)\omega^{4k} - \sum_{k=0}^{(n-2)/4} \delta_{4k+2}(\alpha) r_{4k+2}^{-}(\alpha)\omega^{4k+2}.$$

For its absolute value we have

$$\begin{aligned} |\Delta h(\omega)| &\leq \left|\sum_{k=0}^{n/4} \delta_{4k}(\alpha) r_{4k}^{+}(\alpha)\omega^{4k}\right| + \left|\sum_{k=0}^{(n-2)/4} -\delta_{4k+2}(\alpha) r_{4k+2}^{-}(\alpha)\omega^{4k+2}\right| \\ &\leq \sum_{k=0}^{n/4} \left|\delta_{4k}(\alpha) r_{4k}^{+}(\alpha)\omega^{4k}\right| + \sum_{k=0}^{(n-2)/4} \left|\delta_{4k+2}(\alpha) r_{4k+2}^{-}(\alpha)\omega^{4k+2}\right|. \end{aligned} \tag{2.12}$$

Applying Hölder's inequality one obtains

$$|\Delta h(\omega)| \;\leq\; \left(\sum_{k=0}^{n/2} |\delta_{2k}(\alpha)|^{p}\right)^{\frac{1}{p}} \left(\sum_{k=0}^{n/4} \left(r_{4k}^{+}(\alpha)\omega^{4k}\right)^{q} + \sum_{k=0}^{(n-2)/4} \left(r_{4k+2}^{-}(\alpha)\omega^{4k+2}\right)^{q}\right)^{\frac{1}{q}}. \tag{2.13}$$

Analogically, for $\Delta g(\omega) \geq 0$ we have

$$\Delta g(\omega) \;\leq\; \sum_{k=0}^{(n-1)/4} \delta_{4k+1}(\alpha) r_{4k+1}^{+}(\alpha)\omega^{4k} - \sum_{k=0}^{(n-3)/4} \delta_{4k+3}(\alpha) r_{4k+3}^{-}(\alpha)\omega^{4k+2} \tag{2.14}$$

and

$$|\Delta g(\omega)| \;\leq\; \left(\sum_{k=0}^{(n-1)/2} |\delta_{2k+1}(\alpha)|^{p}\right)^{\frac{1}{p}} \left(\sum_{k=0}^{(n-1)/4} \left(r_{4k+1}^{+}(\alpha)\omega^{4k}\right)^{q} + \sum_{k=0}^{(n-3)/4} \left(r_{4k+3}^{-}(\alpha)\omega^{4k+2}\right)^{q}\right)^{\frac{1}{q}}. \tag{2.15}$$

Substituting (2.9) into (2.13), (2.15) and (2.5) one obtains

$$\left(\frac{|\Delta h(\omega)|}{S_p^{+}(\omega,\alpha)}\right)^{p} + \left(\frac{|\Delta g(\omega)|}{T_p^{+}(\omega,\alpha)}\right)^{p} \leq \sum_{k=0}^{n/2} |\delta_{2k}(\alpha)|^{p} + \sum_{k=0}^{(n-1)/2} |\delta_{2k+1}(\alpha)|^{p} = \sum_{k=0}^{n} |\delta_{k}(\alpha)|^{p} \leq 1^{p}.$$

It means that for $h_0(\omega) \leq 0$ and $g_0(\omega) \leq 0$ the origin is excluded from the value set of the polynomial (2.5) if and only if

$$\left[\left(\frac{|h_0(\omega)|}{S_p^{+}(\omega,\alpha)}\right)^{p} + \left(\frac{|g_0(\omega)|}{T_p^{+}(\omega,\alpha)}\right)^{p}\right]^{\frac{1}{p}} > 1 \tag{2.16}$$

or equivalently if and only if

$$\left[\left(\frac{|h_0(\omega)|}{S_p(\omega,\alpha)}\right)^p + \left(\frac{|g_0(\omega)|}{T_p(\omega,\alpha)}\right)^p\right]^{\frac{1}{p}} > 1.$$

2. $\Delta h(\omega) \le 0, \Delta g(\omega) \le 0$:

For $\Delta h(\omega) \le 0$ we have

$$\Delta h(\omega) \;\ge\; \sum_{k=0}^{n/4} \delta_{4k}(\alpha) r_{4k}^{-}(\alpha)\omega^{4k} - \sum_{k=0}^{(n-2)/4} \delta_{4k+2}(\alpha) r_{4k+2}^{+}(\alpha)\omega^{4k+2} \tag{2.17}$$

or equivalently for its absolute value

$$\begin{aligned} |\Delta h(\omega)| \;&\le\; \left|\sum_{k=0}^{n/4} \delta_{4k}(\alpha) r_{4k}^{-}(\alpha)\omega^{4k}\right| + \left|\sum_{k=0}^{(n-2)/4} -\delta_{4k+2}(\alpha) r_{4k+2}^{+}(\alpha)\omega^{4k+2}\right| \\ &\le\; \sum_{k=0}^{n/4} \left|\delta_{4k}(\alpha) r_{4k}^{-}(\alpha)\omega^{4k}\right| + \sum_{k=0}^{(n-2)/4} \left|\delta_{4k+2}(\alpha) r_{4k+2}^{+}(\alpha)\omega^{4k+2}\right|. \end{aligned} \tag{2.18}$$

Using Hölder's inequality gives

$$|\Delta h(\omega)| \;\le\; \left(\sum_{k=0}^{n/2} |\delta_{2k}(\alpha)|^p\right)^{\frac{1}{p}} \left(\sum_{k=0}^{(n-1)/4} \left(r_{4k+1}^{-}(\alpha)\omega^{4k+1}\right)^q + \sum_{k=0}^{(n-3)/4} \left(r_{4k+3}^{+}(\alpha)\omega^{4k+3}\right)^q\right)^{\frac{1}{q}}. \tag{2.19}$$

Analogically, for $\Delta g(\omega)$ we have

$$\Delta g(\omega) \;\ge\; \sum_{k=0}^{(n-1)/4} \delta_{4k+1}(\alpha) r_{4k+1}^{-}(\alpha)\omega^{4k} - \sum_{k=0}^{(n-3)/4} \delta_{4k+3}(\alpha) r_{4k+3}^{+}(\alpha)\omega^{4k+2} \tag{2.20}$$

and for the absolute value

$$|\Delta g(\omega)| \;\le\; \left(\sum_{k=0}^{(n-1)/2} |\delta_{2k+1}(\alpha)|^p\right)^{\frac{1}{p}} \left(\sum_{k=0}^{(n-1)/4} \left(r_{4k+1}^{-}(\alpha)\omega^{4k}\right)^q + \sum_{k=0}^{(n-3)/4} \left(r_{4k+3}^{+}(\alpha)\omega^{4k+2}\right)^q\right)^{\frac{1}{q}}. \tag{2.21}$$

Substitution of (2.9) into (2.19), (2.21) and (2.5) gives

$$\left(\frac{|\Delta h(\omega)|}{S_p^{-}(\omega,\alpha)}\right)^p + \left(\frac{|\Delta g(\omega)|}{T_p^{-}(\omega,\alpha)}\right)^p \le \sum_{k=0}^{n/2} |\delta_{2k}(\alpha)|^p + \sum_{k=0}^{(n-1)/2} |\delta_{2k+1}(\alpha)|^p = \sum_{k=0}^{n} |\delta_k(\alpha)|^p \le 1$$

or equivalently

$$\left(\frac{|\Delta h(\omega)|}{S_p^-(\omega,\alpha)}\right)^p + \left(\frac{|\Delta g(\omega)|}{T_p^-(\omega,\alpha)}\right)^p \le 1^p. \tag{2.22}$$

It means that for $h_0(\omega) \ge 0$ and $g_0(\omega) \ge 0$ the origin is excluded from the value set of the polynomial (2.5) if and only if

$$\left[\left(\frac{|h_0(\omega)|}{S_p^-(\omega,\alpha)}\right)^p + \left(\frac{|g_0(\omega)|}{T_p^-(\omega,\alpha)}\right)^p\right]^{\frac{1}{p}} > 1 \tag{2.23}$$

or equivalently if and only if

$$\left[\left(\frac{|h_0(\omega)|}{S_p(\omega,\alpha)}\right)^p + \left(\frac{|g_0(\omega)|}{T_p(\omega,\alpha)}\right)^p\right]^{\frac{1}{p}} > 1.$$

Using similar reasoning for $h_0(\omega) \le 0$, $g_0(\omega) \ge 0$ and $h_0(\omega) \ge 0$, $g_0(\omega) \le 0$ one can state that the origin is excluded from the value set of the polynomial (2.5) if and only if

$$\left[\left(\frac{|h_0(\omega)|}{S_p(\omega,\alpha)}\right)^p + \left(\frac{|g_0(\omega)|}{T_p(\omega,\alpha)}\right)^p\right]^{\frac{1}{p}} > 1. \tag{2.24}$$

Since $S_p(\omega,\alpha)$, $T_p(\omega,\alpha)$ are for a fixed ω decreasing functions of α stability of (2.5) is violated just when $\alpha = \underline{\alpha}$.

By substituting $p = \infty$ into (2.24) one arrives to the condition c) of the theorem that completes the proof. □

Hence determination of $\alpha_{\min}$ corresponds to computation of zeros of functions appearing in the condition c) of theorem 2.4 for each $\omega > 0$. Nevertheless, the task can be substantially simplified when triangular membership functions are used.

2.4.3 Triangular membership functions

Assume that all the coefficients $\tilde{q}_k, k = 0,\ldots,n$ of the characteristic polynomial (2.1) are described by (generally nonsymmetric) triangular membership functions with $\mathrm{core}(\tilde{q}_k) = [q_k^-, q_k^+]$ and $\mathrm{supp}(\tilde{q}_k) = q_k^0$ (see Fig. 2.1), i.e.

$$\mu_{\tilde{q}_k}(q_k) = \mathrm{tri}(q_k^-, q_k^0, q_k^+) = \begin{cases} \frac{q_k - q_k^-}{q_k^0 - q_k^-}, & \text{if } q_k^- \le q_k < q_k^0 \\ \frac{q_k^+ - q_k}{q_k^+ - q_k^0}, & \text{if } q_k^0 < q_k \le q_k^+ \\ 0 & \text{otherwise} \end{cases}.$$

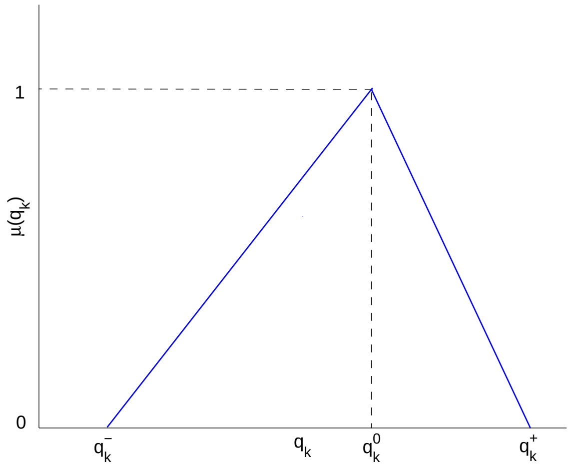

Figure 2.1: Triangular membership function

Then

$$
\begin{aligned}
q_k^-(\alpha) &= (q_k^0 - q_k^-)\alpha + q_k^- \, , \\
q_k^+(\alpha) &= (q_k^0 - q_k^+)\alpha + q_k^+
\end{aligned} \tag{2.25}
$$

and its substituting in (2.4) yields

$$
\begin{aligned}
r_k^-(\alpha) &= (1-\alpha) r_k^-, r_k^+(\alpha) = (1-\alpha) r_k^+, \\
r_k^- &= q_k^0 - q_k^-, \, r_k^+ = q_k^+ - q_k^0, \; k = 0, \ldots, n.
\end{aligned} \tag{2.26}
$$

Substituting (2.25) into (2.9) and theorem 2.4 gives

$$
\alpha_\infty = \max \left\{ \frac{r_n^- - q_n^0}{r_n^-}, 0 \right\}, \tag{2.27}
$$

$$
\alpha_0 = \max \left\{ \frac{r_0^- - q_0^0}{r_0^-}, 0 \right\}, \tag{2.28}
$$

$$
\alpha_\omega = \max \left\{ 1 - \max \left\{ \frac{|h_0(\omega)|}{S_\infty(\omega)}, \frac{|g_0(\omega)|}{T_\infty(\omega)} \right\}, 0 \right\} \tag{2.29}
$$

where

$$
\begin{aligned}
S_\infty(\omega) &= 0.5\left(S_\infty^-(\omega)+S_\infty^+(\omega)\right)+0.5\left(S_\infty^-(\omega)-S_\infty^+(\omega)\right)\operatorname{sgn}\mathrm{h}_0(\omega),\\
S_\infty^+(\omega) &= \sum_{k=0}^{n/2} r_{2k}^+\omega^{2k}+\sum_{k=0}^{(n-2)/4} r_{4k+2}^-\omega^{4k+2},\\
S_\infty^-(\omega) &= \sum_{k=0}^{n/4} r_{4k}^-\omega^{4k}+\sum_{k=0}^{(n-2)/4} r_{4k+2}^+\omega^{4k+2},\\
T_\infty(\omega) &= 0.5\left(T_\infty^-(\omega)+T_\infty^+(\omega)\right)+0.5\left(T_\infty^-(\omega)-T_\infty^+(\omega)\right)\operatorname{sgn}\mathrm{g}_0(\omega),\\
T_\infty^+(\omega) &= \sum_{k=0}^{(n-1)/4} r_{4k+1}^+\omega^{4k}+\sum_{k=0}^{(n-3)/4} r_{4k+3}^-\omega^{4k+2},\\
T_\infty^-(\omega) &= \sum_{k=0}^{(n-1)/4} r_{4k+1}^-\omega^{4k}+\sum_{k=0}^{(n-3)/4} r_{4k+3}^+\omega^{4k+2}.
\end{aligned}
$$

The relation (2.29) comes from the fact that solution of

$$\max\left\{\frac{|h_0(\omega)|}{(1-\alpha)S_\infty(\omega)},\frac{|g_0(\omega)|}{(1-\alpha)T_\infty(\omega)}\right\}=1 \tag{2.30}$$

on $\alpha\in[0,1]$ is given by

$$\alpha_\omega=\min\left\{1-\frac{|h_0(\omega)|}{S_\infty(\omega)},1-\frac{|g_0(\omega)|}{T_\infty(\omega)}\right\}=1-\max\left\{\frac{|h_0(\omega)|}{S_\infty(\omega)},\frac{|g_0(\omega)|}{T_\infty(\omega)}\right\} \tag{2.31}$$

since both arguments in (2.30) are positive increasing functions of $\alpha\in(0,1]$.

Remark 2.4 From (2.31) follows that

$$\underline{\alpha}=\sup_{\omega>0}\alpha_\omega=1-\inf_{\omega>0}\left\{\max\left\{\frac{|h_0(\omega)|}{S_\infty(\omega)},\frac{|g_0(\omega)|}{T_\infty(\omega)}\right\}\right\}=1-\rho. \tag{2.32}$$

The value of ρ can be determined graphically as the half of the maximum length of the side of the square centered in the origin of the complex plane that is not crossed by the frequency plot of $\left(\frac{h_0(\omega)}{S_\infty(\omega)}+\mathrm{j}\frac{g_0(\omega)}{T_\infty(\omega)}\right)$ for $\omega>0$, see example 2.1.

Example 2.1 – Interval fuzzy system

Let the coefficients of a 6-th order polynomial $\tilde{p}(s)=\sum_{k=0}^{n}\tilde{q}_k s^k$ be described by the following triangular membership functions:

$$\tilde{q}_0\approx\mathrm{tri}(0.3,1,1.6);\ \tilde{q}_1\approx\mathrm{tri}(4.5,7,8.5);\ \tilde{q}_2\approx\mathrm{tri}(9,16,21.5);$$

$$\tilde{q}_3\approx\mathrm{tri}(13,19,28);\ \tilde{q}_4\approx\mathrm{tri}(10,14,21.5);\ \tilde{q}_5\approx\mathrm{tri}(3.5,5,7.5);$$

$$\tilde{q}_6\approx\mathrm{tri}(0.6,1,2)\,.$$

The nominal (1-cut) polynomial $p(s,1) = 1 + 7s + 16s^2 + 19s^3 + 14s^4 + 5s^5 + s^6$ is stable. The relations (2.26), (2.27) and (2.28) yield $\alpha_\infty = \max\{-1.5, 0\} = 0$ and $\alpha_0 = \max\{-0.429, 0\} = 0$. From the frequency plot of α_ω (2.29) for $\omega \in [0,4]$ shown in Fig. 2.2 we obtain $\underline{\alpha} = \sup_{\omega>0} \alpha_\omega = 0.527$, therefore $\alpha_{\min} = \max\{\alpha_\infty, \alpha_0, \underline{\alpha}\} = 0.527$.

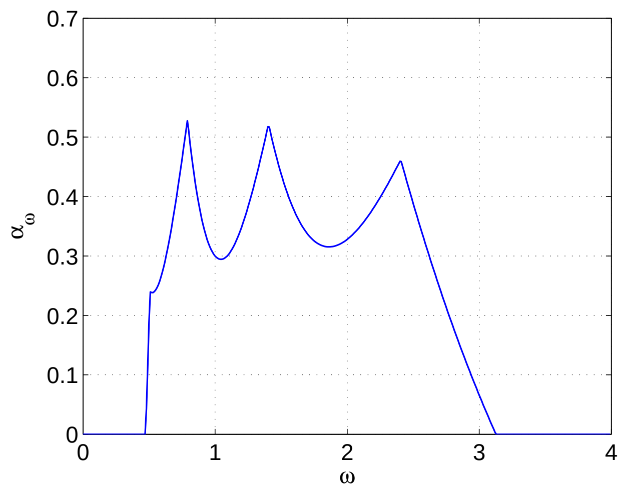

Figure 2.2: Frequency plot of α_ω for $\omega \in [0,4]$

The result is confirmed by the frequency plot of $\left(\frac{h_0(\omega)}{S_\infty(\omega)} + \mathrm{j}\frac{g_0(\omega)}{T_\infty(\omega)}\right)$ for $\omega > 0$ depicted in Fig. 2.3. The half of the maximum length of the side of the square $\rho = 0.473$ and $\underline{\alpha} = 1 - \rho = 0.527$.

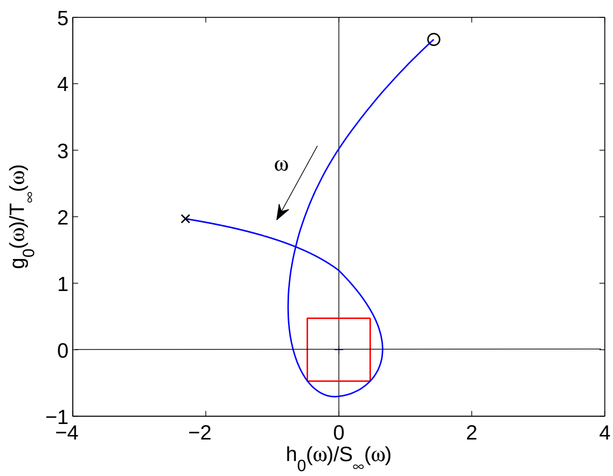

Figure 2.3: Frequency plot of $\left(\frac{h_0(\omega)}{S_\infty(\omega)} + \mathrm{j}\frac{g_0(\omega)}{T_\infty(\omega)}\right)$

2.5 Multidimensional membership function

Until now we have supposed that the parameters of the plant are represented by independent fuzzy numbers. Nevertheless, the parameters of a system or the coefficients of a characteristic polynomial are very often identified using measured input-output data. In such case it is more realistic to characterize the set of parameters by a multidimensional membership function rather than employing fuzzy numbers. For example when utilizing well-known prediction error (PE) identification algorithm ([21, 22, 23, 24]) the coefficients $\mathbf{q} = [q_0, \ldots, q_n]^{\mathrm{T}}$ lie in an ellipsoidal set

$$(\mathbf{q} - \mathbf{q}^0)^{\mathrm{T}} \mathbf{\Gamma} (\mathbf{q} - \mathbf{q}^0) \leq 1 \tag{2.33}$$

where $\mathbf{\Gamma}$ is a positive definite matrix and the vector $\mathbf{q}^0 = [q_0^0, \ldots, q_n^0]^{\mathrm{T}}$ is the parameter nominal value. When performing more sets of measurements each with different confidence level α resulting in $\mathbf{\Gamma}(\alpha)$ it is reasonable, in order to aggregate the knowledge obtained from each of them, to characterize the coefficients by a fuzzy set described by the α-cuts

$$[Q]_\alpha = \left\{ \mathbf{q} : (\mathbf{q} - \mathbf{q}^0)^{\mathrm{T}} \mathbf{\Gamma}(\alpha) (\mathbf{q} - \mathbf{q}^0) \leq 1 \right\} \tag{2.34}$$

where the confidence level α indicates the belief in the experiment the measured data were obtained by. The natural question arises what minimum confidence level $\alpha_{\min}$ is necessary so that the α-cut polynomial (family of polynomials)

$$[\tilde{p}(s)]_{\alpha_{\min}} = q_0 + q_1 s + \cdots + q_n s^n; \quad \mathbf{q} \in [Q]_{\alpha_{\min}} \tag{2.35}$$

remains stable.

2.5.1 Problem formulation

In the sequel we will consider polynomial $\tilde{p}(s)$ defined by its α-cut representation

$$\begin{aligned} [\tilde{p}(s)]_\alpha &= q_0 + q_1 s + \cdots + q_n s^n, \mathbf{q} \in [Q]_\alpha, \\ \mathbf{q} &= [q_0, \ldots, q_n]^{\mathrm{T}}, q_k \in \Re, k = 0, \ldots, n \end{aligned} \tag{2.36}$$

with fuzzy set Q characterized by the α-cuts

$$\begin{aligned} [Q]_\alpha &= \{ \mathbf{q} : \mu_{\tilde{\mathbf{q}}}(\mathbf{q}) \geq \alpha \} \\ &= \left\{ \mathbf{q} : (\mathbf{q} - \mathbf{q}^0)^{\mathrm{T}} \mathbf{\Gamma}(\alpha) (\mathbf{q} - \mathbf{q}^0) \leq 1 \right\} \end{aligned} \tag{2.37}$$

where $\mathbf{q}^0 = [q_0^0, \ldots, q_n^0]^{\mathrm{T}}$ is a nominal point and $\boldsymbol{\Gamma}(\alpha)$ is $(n+1) \times (n+1)$ square diagonal matrix

$$\boldsymbol{\Gamma}(\alpha) = \begin{bmatrix} \frac{1}{\gamma_0^2(\alpha)} & & & 0 \\ & \frac{1}{\gamma_1^2(\alpha)} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\gamma_n^2(\alpha)} \end{bmatrix} \tag{2.38}$$

with

$$\gamma_k(\alpha) = \begin{cases} \gamma_k^+(\alpha) & \text{for} \quad q_k \geq q_k^0 \\ \gamma_k^-(\alpha) & \text{for} \quad q_k < q_k^0, k = 0, \ldots, n \end{cases} \tag{2.39}$$

where $\gamma_k^+(\alpha)$ and $\gamma_k^-(\alpha)$ are nonnegative decreasing functions defined for $0 \leq \alpha \leq 1$.

Let us note that for any $0 \leq \alpha \leq 1$ the corresponding coefficient space of the family of polynomials (2.36) is a non-symmetric axes-parallel hyperellipsoid with the lengths of the semiaxes given by $\gamma_k^-(\alpha)$ and $\gamma_k^+(\alpha)$ for coefficients lying below and above their nominal values, respectively. Non-symmetricity of the hyperellipsoid makes it possible to consider the cases when the nominal point corresponding to most often operating conditions does not lie in the middle of measured data.

Let us suppose that the most confident (1-cut) uncertain polynomial $[\tilde{p}(s)]_{\alpha=1}$ is robustly Hurwitz stable. The task is to find stability margin of the polynomial $\tilde{p}(s)$, i.e. minimum confidence level $\alpha_{\min} \in [0, 1]$ such that uncertain polynomial $[\tilde{p}(s)]_\alpha$ (2.35) is stable for $\alpha > \alpha_{\min}$ and unstable for $\alpha \leq \alpha_{\min}$.

Let us observe that the uncertain polynomial (2.36) can be written as

$$\begin{aligned} [\tilde{p}(s)]_\alpha &= p(s, A) = q_0 + q_1 s + \cdots + q_n s^n, \\ A &= [Q]_\alpha = \left\{ \mathbf{q} : \sum_{k=0}^{n} \left| \frac{q_k - q_k^0}{\gamma_k(\alpha)} \right|^2 \leq 1 \right\}. \end{aligned} \tag{2.40}$$

By letting $p = 2$ in the proof of theorem 2.4 we can find stability margin of $\tilde{p}(s)$ by the following modification.

Theorem 2.5: *Denote by α_∞ and α_0 the solutions of*
a) $q_n^0 = \gamma_n^-(\alpha)$ *,*
b) $q_0^0 = \gamma_0^-(\alpha)$*,*
respectively, and by α_ω the solutions of

c) $\left(\frac{h_0(\omega)}{S_2(\omega,\alpha)}\right)^2 + \left(\frac{g_0(\omega)}{T_2(\omega,\alpha)}\right)^2 = 1$ *with respect to* α *for each* $\omega > 0$,
on the interval $\alpha \in [0, 1]$. *Assign zero in the case that a solution does not exist.*

Then

$$\alpha_{\min} = \max\{\alpha_\infty, \alpha_0, \underline{\alpha}\} \tag{2.41}$$

where $\underline{\alpha} = \sup_{\omega>0} \alpha_\omega$.

Example 2.2 – Aggregation of more sets of measurements

Assume that 5 sets of measurements with different confidence level were provided to identify characteristic polynomial of a 6-th order system by using the prediction error algorithm. The obtained results are summarized in table 2.1.

Table 2.1: Measured data for example 2.2

α	γ_6^+	γ_5^+	γ_4^+	γ_3^+	γ_2^+	γ_1^+	γ_0^+
0	0.6000	8.560	12.98	45.95	117.4	143.4	196.9
0.3	0.5781	8.092	12.21	42.65	106.5	138.2	182.1
0.5	0.5126	7.574	10.03	37.05	95.81	117.9	164.4
0.7	0.3674	5.436	7.288	24.57	74.45	83.08	115.1
1	0.3278	4.582	6.193	21.97	56.71	74.74	95.49

α	γ_6^-	γ_5^-	γ_4^-	γ_3^-	γ_2^-	γ_1^-	γ_0^-
0	0.384	5.247	24.47	57.36	104.5	130.4	149.4
0.3	0.3743	5.723	23.34	55.28	92.52	123.7	148.7
0.5	0.3357	4.561	17.09	53.25	82.46	110.4	133.3
0.7	0.2467	3.576	7.788	22.57	77.45	81.08	118.1
1	0.2327	2.642	10.23	33.15	48.27	63.72	73.73

The nominal polynomial was identified as

$$p^0(s) = s^6 + 14s^5 + 80.25s^4 + 251.25s^3 + 502.25s^2 + 667.25s + 433.5.$$

In order to aggregate the knowledge obtained from all measurements we describe the characteristic polynomial by α-cuts characterized by the multidimensional membership function (2.37). Let us use polynomic interpolation and least square method to obtain the functions $\gamma_k^+(\alpha)$ and $\gamma_k^-(\alpha)$. The coefficients of the second order polynomials, $\gamma(\alpha) = c_2\alpha^2 + c_1\alpha + c_0$, can be found in table 2.2.

Table 2.2: Polynomic interpolation of $\gamma_k^+(\alpha)$ and $\gamma_k^-(\alpha)$ for example 2.2

	γ_6^+	γ_5^+	γ_4^+	γ_3^+	γ_2^+	γ_1^+	γ_0^+
c_2	-0.1148	-2.391	-1.249	-5.010	-25.20	-20.73	-40.66
c_1	-0.1925	-1.953	-6.298	-21.89	-38.16	-57.46	-69.86
c_0	0.6155	8.701	13.34	47.22	118.5	147.8	200.6
α	γ_6^-	γ_5^-	γ_4^-	γ_3^-	γ_2^-	γ_1^-	γ_0^-
c_2	-0.0574	-1.239	-3.771	-26.87	-33.02	-38.70	-52.77
c_1	-0.1170	-2.161	-11.45	2.312	-20.64	-33.48	-25.77
c_0	0.3942	5.944	25.09	57.40	103.4	132.7	150.5

The characteristic polynomial is then represented by the α-cuts (2.36) characterized by (2.37) with $\mathbf{q}^0 = [433.5,\ 667.25,\ 502.25,\ 251.25,\ 80.25,\ 14,\ 1]^{\mathrm{T}}$ and $\gamma_k^+(\alpha), \gamma_k^-(\alpha), k = 0, \ldots, n$ given by table 2.2. Let us note that the 1-cut polynomial $[\tilde{p}(s)]_{\alpha=1}$ is an uncertain polynomial and does not equal to the nominal polynomial $p^0(s)$.

From the frequency plot of α_ω according to the condition c) of theorem 2.5 depicted in Fig. 2.4 we have $\underline{\alpha} = \sup_{\omega>0} \alpha_\omega = 0.8182$. From the conditions a) and b) of theorem 2.5 one obtains $\alpha_\infty = 0.0822$ and $\alpha_0 = 0.0404$ and using (2.41) the minimum confidence level guaranteeing stability $\alpha_{\min} = \max\{\alpha_\infty, \alpha_0, \underline{\alpha}\} = 0.8182$.

Example 2.3 – Multidimensional membership function

Let us suppose that the functions $\gamma_k^+(\alpha)$ and $\gamma_k^-(\alpha)$ are given as

$$\begin{aligned} \gamma_k^+(\alpha) &= \gamma_k^+ \cdot (1-\alpha),\ \gamma_k^-(\alpha) = \gamma_k^- \cdot (1-\alpha), \\ \gamma_k^+ &> 0, \gamma_k^- > 0, k = 0, \ldots, n. \end{aligned} \tag{2.42}$$

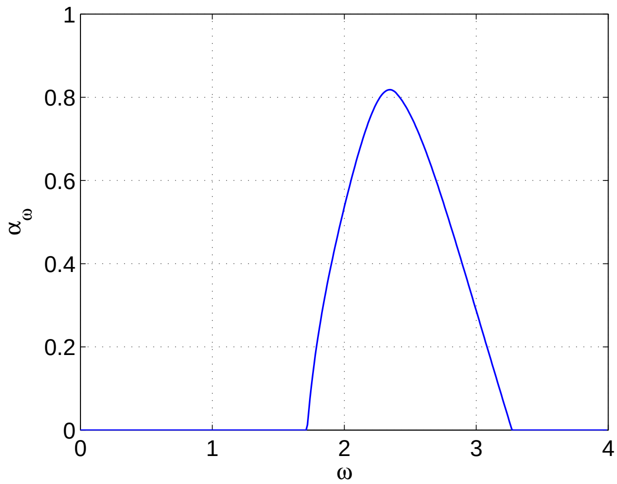

Figure 2.4: Frequency plot of α_ω for example 2.2 for $\omega \in [0, 4]$

Then substituting (2.42) into (2.9) gives

$$\alpha_\infty = \max\left\{\frac{q_n^0 - \gamma_n^-}{\gamma_n^-}, 0\right\}, \tag{2.43}$$

$$\alpha_0 = \max\left\{\frac{q_0^0 - \gamma_0^-}{\gamma_0^-}, 0\right\}, \tag{2.44}$$

$$\alpha_\omega = \max\left\{1 - \left(\left(\frac{h_0(\omega)}{S_2(\omega)}\right)^2 + \left(\frac{g_0(\omega)}{T_2(\omega)}\right)^2\right)^{\frac{1}{2}}, 0\right\} \tag{2.45}$$

where

$$\begin{aligned}
S_2(\omega) &= 0.5\left(S_2^-(\omega) + S_2^+(\omega)\right) + 0.5\left(S_2^-(\omega) - S_2^+(\omega)\right)\operatorname{sgn}\mathrm{h}_0(\omega),\\
S_2^+(\omega) &= \left(\sum_{k=0}^{n/2}\left(\gamma_{2k}^+\omega^{2k}\right)^2 + \sum_{k=0}^{(n-2)/4}\left(\gamma_{4k+2}^-\omega^{4k+2}\right)^2\right)^{\frac{1}{2}},\\
S_2^-(\omega) &= \left(\sum_{k=0}^{n/4}\left(\gamma_{4k}^-\omega^{4k}\right)^2 + \sum_{k=0}^{(n-2)/4}\left(\gamma_{4k+2}^+\omega^{4k+2}\right)^2\right)^{\frac{1}{2}},\\
T_2(\omega) &= 0.5\left(T_2^-(\omega) + T_2^+(\omega)\right) + 0.5\left(T_2^-(\omega) - T_2^+(\omega)\right)\operatorname{sgn}\mathrm{g}_0(\omega),\\
T_2^+(\omega) &= \left(\sum_{k=0}^{(n-1)/4}\left(\gamma_{4k+1}^+\omega^{4k}\right)^2 + \sum_{k=0}^{(n-3)/4}\left(\gamma_{4k+3}^-\omega^{4k+2}\right)^2\right)^{\frac{1}{2}},\\
T_2^-(\omega) &= \left(\sum_{k=0}^{(n-1)/4}\left(\gamma_{4k+1}^-\omega^{4k}\right)^2 + \sum_{k=0}^{(n-3)/4}\left(\gamma_{4k+3}^+\omega^{4k+2}\right)^2\right).
\end{aligned}$$

Particularly, let the coefficients of a 6-th order polynomial (2.36) be characterized by (2.37)–(2.39) specified by (2.42) with the following values:

$$
\begin{aligned}
\mathbf{q^0} &= [q_0^0, q_1^0, q_2^0, q_3^0, q_4^0, q_5^0, q_6^0]^{\mathrm{T}} \\
&= [433.5,\ 667.25,\ 502.25,\ 251.25,\ 80.25,\ 14,\ 1]^{\mathrm{T}}, \\
\gamma^+ &= [\gamma_0^+, \gamma_1^+, \gamma_2^+, \gamma_3^+, \gamma_4^+, \gamma_5^+, \gamma_6^+]^{\mathrm{T}} \\
&= [196.98,\ 143.48,\ 117.42,\ 45.95,\ 12.98,\ 8.56,\ 0.60]^{\mathrm{T}}, \\
\gamma^- &= [\gamma_0^-, \gamma_1^-, \gamma_2^-, \gamma_3^-, \gamma_4^-, \gamma_5^-, \gamma_6^-]^{\mathrm{T}} \\
&= [153.40,\ 133.44,\ 100.55,\ 60.30,\ 22.47,\ 5.60,\ 0.40]^{\mathrm{T}}.
\end{aligned}
$$

The 1-cut polynomial (identical to nominal polynomial) $[\tilde{p}(s)]_{\alpha=1} = \sum_{k=0}^{6} q_k^0 s^k = 433.5 + 667.25s + 502.25s^2 + 251.25s^3 + 80.25s^4 + 14s^5 + s^6$ is Hurwitz stable. From the frequency plot of α_ω according to (2.45) depicted in Fig. 2.5 we have $\underline{\alpha} = \sup_{\omega>0} \alpha_\omega = 0.3384$. Using (2.43) and (2.44) one obtains $\alpha_\infty = \max\{-1.5, 0\} = 0$, $\alpha_0 = \max\{-1.826, 0\} = 0$ and using (2.41) the minimum confidence level guaranteeing stability $\alpha_{\min} = \max\{\alpha_\infty, \alpha_0, \underline{\alpha}\} = 0.3384$.

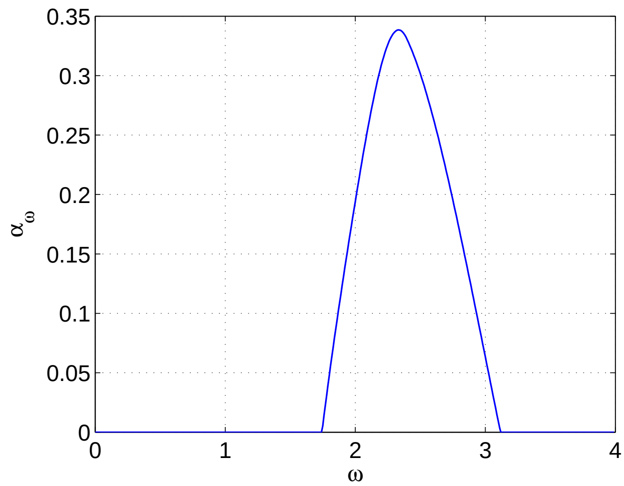

Figure 2.5: Frequency plot of α_ω for example 2.3 for $\omega \in [0, 4]$

Analogically to the interval fuzzy systems the value of $\underline{\alpha}$ can be determined graphically as one minus the radius of the circle centered in the origin of the complex plane that is not crossed by the frequency plot of $\left(\frac{h_0(\omega)}{S_2(\omega)} + \mathrm{j}\frac{g_0(\omega)}{T_2(\omega)}\right)$ for $\omega > 0$.

The frequency plot is shown in Fig. 2.6. Radius of the circle $\rho = 0.66$ implies $\underline{\alpha} = 1 - \rho = 0.34$.

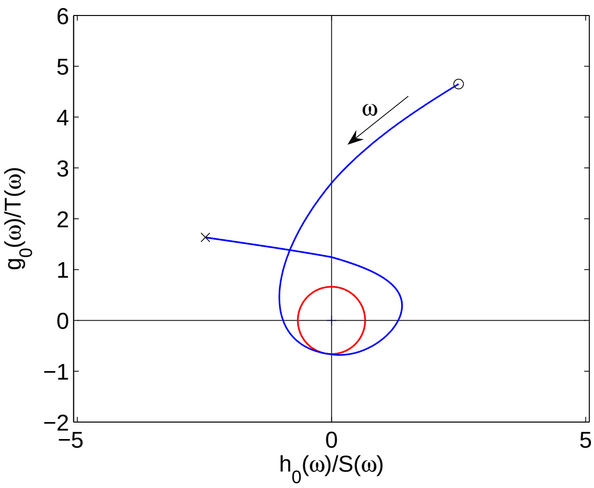

Figure 2.6: Frequency plot of $\left(\frac{h_0(\omega)}{S_2(\omega)} + \mathrm{j}\frac{g_0(\omega)}{T_2(\omega)}\right)$

Chapter 3

Fuzzy linear systems with linear parameter dependency

3.1 Introduction

In previous chapter we have dealt with the problem of determination of minimum confidence level preserving stability of interval fuzzy systems, i.e. the case when each coefficient of the characteristic polynomial depends on one parameter. Unfortunately, it is a rare exception when the coefficients of the characteristic polynomial vary independently. More typically the parameters enter in the characteristic polynomial in linear, multilinear or polynomic fashion. This chapter is concerned with systems with linear affine dependency of coefficients of characteristic polynomial on system parameters that are described by fuzzy functions. Such consideration includes closed loop control of an interval fuzzy system by a fixed controller.

Minimum confidence level preserving stability corresponds to stability margin in robust control theory of systems with parametric uncertainty. Such a problem is in the case of linear parameter dependency usually treated using linear programming techniques [25] or looking for parameter-dependent quadratic Lyapunov functions that results in solving linear matrix inequalities [26], [27]. However when considering varying parameter weights that correspond to arbitrary shape of membership functions a bisection method has to be used and those methods become inefficient.

We adopt the geometric approach that transforms a multidimensional problem to one parameter grid and a test of two-dimensional sets. In order to find a stability margin generalization of the Tsypkin-Polyak plot [28] will be used that allows to deal with nonlinear nature of the problem in the case when arbitrary shape of membership functions describing fuzzy numbers is considered.

3.2 Problem statement

In the sequel we will consider the polynomial

$$\tilde{D}(s) = \tilde{d}_0 + \tilde{d}_1 s + \cdots + \tilde{d}_n s^n \tag{3.1}$$

where the coefficients $\tilde{d}_i, i = 0, \ldots, n$ are supposed to be linear affine functions of the parameters $\tilde{q}_k, k = 1, \ldots, l$, i.e.

$$\tilde{d}_i = \beta_i + \sum_{k=1}^{l} \gamma_{ik} \tilde{q}_k, \quad \beta_i, \gamma_{ik} \in \Re\,. \tag{3.2}$$

The parameters $\tilde{q}_k, k = 1, \ldots, l$ are supposed to be described by fuzzy numbers characterized by arbitrary convex, normal, continuous membership functions $\alpha = \mu_{\tilde{q}_k}(q_k)$ sharing common confidence level α with one-element cores core$\{\tilde{q}_k\} = q_k^0$ and bounded supports supp$\{\tilde{q}_k\} = [q_k^-, q_k^+]$. Then there exist the monotonic increasing and monotonic decreasing functions $q_k^-(\alpha)$ and $q_k^+(\alpha)$, respectively, defined for $\alpha \in [0, 1]$ as

$$\begin{aligned} q_k^-(\alpha) &= \mu_{\tilde{q}_k}^{-1}(\alpha) \quad \text{for} \quad q_k^- \leq q_k^-(\alpha) \leq q_k^0 \\ q_k^+(\alpha) &= \mu_{\tilde{q}_k}^{-1}(\alpha) \quad \text{for} \quad q_k^0 \leq q_k^+(\alpha) \leq q_k^+ \end{aligned} \tag{3.3}$$

that constitute the α-cut representation of parameter $\tilde{q}_k$, $[\tilde{q}_k]_\alpha = [q_k^-(\alpha), q_k^+(\alpha)]$. The functions $q_k^-(\alpha)$ and $q_k^+(\alpha)$ satisfy $q_k^-(0) = q_k^-$, $q_k^+(0) = q_k^+$, $q_k^-(1) = q_k^+(1) = q_k^0$, $q_k^-(\alpha) \leq q_k^+(\alpha)$ and in compliance with standard fuzzy arithmetic [6] (based on interval arithmetic) characterize the α-cut representation of polynomial (3.1)

$$[\tilde{D}(s)]_\alpha = D(s, \alpha) = d_0(\alpha) + d_1(\alpha)s + \cdots + d_n(\alpha)s^n \tag{3.4}$$

where

$$d_i(\alpha) = \beta_i + \sum_{k=1}^{l} \gamma_{ik} q_k,\ i = 0, \ldots, n\,,\ q_k \in [q_k^-(\alpha), q_k^+(\alpha)]\,. \tag{3.5}$$

Let us suppose that the nominal (1-cut) polynomial corresponding to maximum confidence level $D(s, 1) = \sum_{i=0}^{n} d_i^0 s^i, d_i^0 = \beta_i + \sum_{k=1}^{l} \gamma_{ik} q_k^0$ is stable. We are looking for minimum confidence level preserving stability of polynomial (3.1), i.e. the value $\alpha_{\min} \in [0,1]$ such that linear interval polynomial (3.4) is stable for $\alpha > \alpha_{\min}$ and unstable for $\alpha \leq \alpha_{\min}$.

3.2.1 Main result

The polynomial (3.4) can be written as

$$D(s,\alpha) = A(s) + \sum_{k=1}^{l} r_k B_k(s),\ r_k \in [r_k^-(\alpha), r_k^+(\alpha)] \tag{3.6}$$

where

$$\begin{aligned} A(s) &= d_0^0 + d_1^0 s + \cdots + d_n^0 s^n, \quad d_i^0 = \beta_i + \sum_{k=1}^{l} \gamma_{ik} q_k^0, \quad i = 0,\ldots,n\,, \\ B_k(s) &= \gamma_{0k} + \gamma_{1k} s + \cdots + \gamma_{nk} s^n\,, \\ r_k^-(\alpha) &= q_k^-(\alpha) - q_k^0, \quad r_k^+(\alpha) = q_k^+(\alpha) - q_k^0, \quad k = 1,\ldots,l\,. \end{aligned} \tag{3.7}$$

Let us examine the value set of polynomial family (3.6) in some point $s = \mathrm{j}\omega^*$,

$$D(\mathrm{j}\omega^*,\alpha) = D(\alpha) = A(\mathrm{j}\omega^*) + \sum_{k=1}^{l} r_k B_k(\mathrm{j}\omega^*),\ r_k \in [r_k^-(\alpha), r_k^+(\alpha)]\,. \tag{3.8}$$

Denote by $\underline{\alpha}(\omega^*)$ the maximum value of α, $\alpha \in [0,1]$, such that zero is included in the value set (3.8):

$$\underline{\alpha}(\omega^*) := \sup\{0 \leq \alpha \leq 1 : 0 \in D(\mathrm{j}\omega^*,\alpha)\}. \tag{3.9}$$

In the case a limiting value is never achieved, set $\underline{\alpha}(\omega^*) = 0$. Since $D(s,\alpha)$ is stable for $\alpha = 1$ then $0 \notin D(\mathrm{j}\omega^*, 1)$ and $\underline{\alpha}(\omega^*) < 1$ for $0 \leq \omega^* < \infty$. If we define

$$\underline{\alpha}_\omega = \sup_{\omega^* \geq 0} \underline{\alpha}(\omega^*) \tag{3.10}$$

then using zero exclusion theorem

$$\alpha_{\min} = \max\{\underline{\alpha}_\omega, \alpha_\infty\} \tag{3.11}$$

where the value α_∞ corresponds to degree drop of polynomial (3.4) and can be determined as a solution of

$$\textstyle\sum_{k=1}^{l} r_k^*(\alpha_\infty)\gamma_{nk} = -d_n^0 \tag{3.12}$$

with

$$r_k^*(\cdot) = \begin{cases} r_k^-(\cdot), & \text{if } \mathrm{sign}(\gamma_{nk}) = \text{if } \mathrm{sign}(d_n^0) \\ r_k^+(\cdot), & \text{if } \mathrm{sign}(\gamma_{nk}) \neq \text{if } \mathrm{sign}(d_n^0) \end{cases}$$

or $\alpha_\infty = 0$ if a solution of (3.12) does not exist. Since $r_k^-(\cdot)$ and $r_k^+(\cdot)$ are strictly monotonic the solution of (3.12), if exists, is unique.

We will now try to answer the question how to determine the value $\underline{\alpha}(\omega^*)$ for $0 \leq \omega^* < \infty$.

Theorem 3.1: *Denote*

$$\begin{aligned} A &= A(\mathrm{j}\omega^*) = |A|e^{\mathrm{j}\theta}\,, \\ B_k &= B_k(\mathrm{j}\omega^*) = |B_k|e^{\mathrm{j}\phi_k}, k = 1, \ldots, l. \end{aligned} \tag{3.13}$$

Then

$$\underline{\alpha}(\omega^*) = \begin{cases} \min_{1\leq k\leq l}\left\{\alpha_k \,:\, |A||\sin(\theta-\phi_k)| = \sum_{i=1}^{l}|r_i^*(\alpha_k)||B_i||\sin(\phi_i-\phi_k)|\right\}, \\ 0, \quad \text{if } \alpha_k \text{ does not exist } \forall k = 1, \ldots, l \end{cases}$$
$$\text{if } \sin(\phi_i - \phi_k) \neq 0 \text{ for some } i, k \tag{3.14}$$

where

$$r_i^*(\cdot) = \begin{cases} r_i^-(\cdot), & \text{if } \mathrm{sign}\left(\sin(\phi_i-\phi_k)\right) = \mathrm{sign}\left(\sin(\theta-\phi_k)\right) \\ r_i^+(\cdot), & \text{if } \mathrm{sign}\left(\sin(\phi_i-\phi_k)\right) \neq \mathrm{sign}\left(\sin(\theta-\phi_k)\right) \end{cases},$$

$$\underline{\alpha}(\omega^*) = \begin{cases} \alpha_0 \,:\, |A| = \sum_{i=1}^{l}|r_i^*(\alpha_0)||B_i| \\ 0, \quad \text{if } \alpha_0 \text{ does not exist} \end{cases}, \tag{3.15}$$
$$\text{if } \sin(\phi_i-\phi_k) = 0 \text{ and } \sin(\theta-\phi_k) = 0\ \forall i, k$$

where

$$r_i^*(\cdot) = \begin{cases} r_i^-(\cdot), & \text{if } \theta = \phi_k \\ r_i^+(\cdot), & \text{if } \theta = -\phi_k \end{cases}$$

and

$$\underline{\alpha}(\omega^*) = 0\,, \tag{3.16}$$
$$\text{if } \sin(\phi_i-\phi_k) = 0 \text{ and } \sin(\theta-\phi_k) \neq 0\ \forall i, k.$$

Proof: The value set (3.8) is a parallelogram with $2l$ vertices, in which there are l pairs of edges parallel with $B_k = B_k(\mathrm{j}\omega^*), k = 1, \ldots, l$, see [29], and thus can be described by $2l$ linear inequalities. The set of all $2l$ vertices of $D(\alpha)$ is a subset of the set V_c of 2^l vertex candidates, $V_\mathrm{c} \subset D(\alpha)$, that can be expressed in terms of the coordinates in complex plane as

$$V_\mathrm{c} = \left\{ \left[|A|\cos\theta + \sum_{i=1}^{l} r_i^*(\alpha)|B_i|\cos\phi_i, |A|\sin\theta + \sum_{i=1}^{l} r_i^*(\alpha)|B_i|\sin\phi_i \right] \right\} \tag{3.17}$$

where $r_i^*(\alpha) = r_i^-(\alpha)$ or $r_i^*(\alpha) = r_i^+(\alpha), i = 1, \ldots, l$.

The set of all $2l$ inequalities describing $D(\alpha)$ is a subset of inequalities candidates I_c that are associated to the lines (by substituting the inequality sign with the equal sign) going through a point $V \in V_\mathrm{c}$ in one of the directions of $B_k, k = 1, \ldots, l$, i.e.

$$I_\mathrm{c} = \{(|B_k|\sin\phi_k)\cdot x - (|B_k|\cos\phi_k)\cdot y + c_k(\mathbf{r}(\alpha)) \geq (\leq)\, 0, k = 1, \ldots l\} \tag{3.18}$$

where x and y stands for real and imaginary part of a point in complex plane, respectively, and

$$\begin{aligned} c_k(\mathbf{r}(\alpha)) &= (-|B_k|\sin\phi_k)\left(|A|\cos\theta + \sum_{i=1}^{l} r_i^*(\alpha)|B_i|\cos\phi_i\right) \\ &\quad +(|B_k|\cos\phi_k)\left(|A|\sin\theta + \sum_{i=1}^{l} r_i^*(\alpha)|B_i|\sin\phi_i\right), k = 1, \ldots, l, \\ \mathbf{r}(\alpha) &= [r_1^*(\alpha), \ldots, r_l^*(\alpha)],\ r_i^*(\alpha) = r_i^-(\alpha) \text{ or } r_i^+(\alpha), i = 1, \ldots, l. \end{aligned} \tag{3.19}$$

The inequality signs depend on the choice of k and $r_i^*(\alpha)$ and can be determined by relative position of A. Let us note that because of duplicity only half of inequalities in I_c are different. Those lines associated to inequalities from I_c that do not correspond to the edges of $D(\alpha)$ intersect $D(\alpha)$ since they connect two points of V_c and therefore give too restrictive constraints.

Thus the sets E_k containing only 2 inequalities of interest for each $k = 1, \ldots, l$ associated to the 2 edges of $D(\alpha)$ in the direction B_k are formed by the least restrictive constraints that are determined by the extreme courses of functions $c_k(\mathbf{r}(\alpha))$ given by

$$\begin{aligned} c_{k\,\min}(\alpha) &= \min_{\mathbf{r}(\alpha)} c_k(\mathbf{r}(\alpha)), \\ c_{k\,\max}(\alpha) &= \max_{\mathbf{r}(\alpha)} c_k(\mathbf{r}(\alpha)), k = 1, \ldots, l, \end{aligned} \tag{3.20}$$

i.e.

$$\begin{aligned} E_k &= \{(|B_k|\sin\phi_k)\cdot x - (|B_k|\cos\phi_k)\cdot y + c_{k\,\min}(\alpha) \geq (\leq)\, 0, \\ &\quad (|B_k|\sin\phi_k)\cdot x - (|B_k|\cos\phi_k)\cdot y + c_{k\,\max}(\alpha) \geq (\leq)\, 0, k = 1,\dots l\} \end{aligned} \tag{3.21}$$

where the signs of inequality depend on k and α and are given by relative position of A.

The functions $c_k(\mathbf{r}(\alpha))$ can be written as

$$\begin{aligned} c_k(\mathbf{r}(\alpha)) &= |B_k|\left(|A|\,(\sin\theta\cos\phi_k - \sin\phi_k\cos\theta) + \sum_{i=1}^{l} r_i^*(\alpha)|B_i|(\sin\phi_i\cos\phi_k - \sin\phi_k\cos\phi_i)\right) \\ &= |B_k|\left(|A|\sin(\theta-\phi_k) + \sum_{i=1}^{l} r_i^*(\alpha)|B_i|\sin(\phi_i-\phi_k)\right), k = 1,\dots l. \end{aligned} \tag{3.22}$$

Since $r_i^+(\alpha) \geq 0$ and $r_i^-(\alpha) \leq 0$ for $\alpha \in [0,1]$ the $c_k(\mathbf{r}(\alpha)), k = 1,\dots,l$ achieve its minimum value $c_{k\,\min}(\alpha)$ for any $\alpha \in [0,1]$ for

$$r_i^*(\alpha) = \begin{cases} r_i^-(\alpha), & \text{if } \sin(\phi_i-\phi_k) \geq 0 \\ r_i^+(\alpha), & \text{if } \sin(\phi_i-\phi_k) < 0 \end{cases} \tag{3.23}$$

and its maximum value $c_{k\,\max}(\alpha)$ for any $\alpha \in [0,1]$ for

$$r_i^*(\alpha) = \begin{cases} r_i^-(\alpha), & \text{if } \sin(\phi_i-\phi_k) \leq 0 \\ r_i^+(\alpha), & \text{if } \sin(\phi_i-\phi_k) > 0. \end{cases} \tag{3.24}$$

Since $r_i^-(\alpha)$ and $r_i^+(\alpha)$ are strictly increasing and strictly decreasing functions, respectively, then $c_{k\,\min}(\alpha)$ and $c_{k\,\max}(\alpha)$ are strictly increasing and strictly decreasing functions, respectively, too. Moreover, $c_{k\,\max}(1) = c_{k\,\min}(1) = |B_k||A|\sin(\theta-\phi_k)$.

We are looking for minimum value $\alpha = \underline{\alpha} \in [0,1]$ such that $0 \in D(\underline{\alpha})$, i.e. the point $(x,y) = (0,0)$ meets both inequalities in $E_k\,\forall k = 1,\dots,l$. If we denote by $\alpha_k, k = 1,\dots,l$ the minimum value $\alpha \in [0,1]$ such that both inequalities in E_k are met for $(x,y) = (0,0)$ then

$$\underline{\alpha} = \begin{cases} \min_k \alpha_k & \\ 0, & \text{if } \alpha_k \text{ does not exist } \forall k = 1,\dots,l. \end{cases} \tag{3.25}$$

Substituting $x = 0, y = 0$ into (3.21) one obtains

$$(c_{k\,\min}(\alpha) \leq 0 \quad \wedge \quad c_{k\,\max}(\alpha) \leq 0) \quad \vee \tag{3.26}$$

$$(c_{k\,\min}(\alpha) \leq 0 \quad \wedge \quad c_{k\,\max}(\alpha) \geq 0) \quad \vee \tag{3.27}$$

$$(c_{k\,\min}(\alpha) \geq 0 \quad \wedge \quad c_{k\,\max}(\alpha) \geq 0). \tag{3.28}$$

The value α_k will be obtained as minimum solution of one of the pair of inequalities (3.26) – (3.28) (one of them does not correspond to an edge, the other one has no solution). Since both $c_{k\,\min}(\alpha)$ and $c_{k\,\max}(\alpha)$ are strictly monotonic one can obtain α_k by solving

$$c_{k\,\min}(\alpha_k) = 0 \quad \vee \quad c_{k\,\max}(\alpha_k) = 0. \tag{3.29}$$

Since $c_{k\,\min}(\cdot)$ is strictly increasing the equation $c_{k\,\min}(\alpha_k) = 0$ has a solution only if $c_{k\,\min}(1) = |B_k||A|\sin(\theta - \phi_k) \geq 0$, since $c_{k\,\max}(\cdot)$ is strictly decreasing the equation $c_{k\,\max}(\alpha_k) = 0$ has a solution only if $c_{k\,\max}(1) = |B_k||A|\sin(\theta - \phi_k) \leq 0$.

Applying this fact for solving the first equation from (3.29) and using (3.22) and (3.24) the α_k, $k = 1, \ldots, l$, if exists, can be obtained as a solution of

$$|A||\sin(\theta - \phi_k)| = \sum_{i=1}^{l} |r_i^*(\alpha_k)||B_i||\sin(\phi_i - \phi_k)| \tag{3.30}$$

where

$$r_i^*(\alpha) \quad = \quad \begin{cases} r_i^-(\alpha), & \text{if } \sin(\theta - \phi_k) \geq 0 \text{ and } \sin(\phi_i - \phi_k) \geq 0 \\ r_i^+(\alpha), & \text{if } \sin(\theta - \phi_k) \geq 0 \text{ and } \sin(\phi_i - \phi_k) < 0. \end{cases} \tag{3.31}$$

This result combined with the analogical solution of the second equation from (3.29) and (3.25) form (3.14).

The relations (3.15) and (3.16) solve the case when the set $D(\alpha)$ degenerates to a line which is and is not parallel to the vector A, respectively. □

In order to determine stability margin $\alpha_{\min}$ by (3.11) we grid the frequency, compute $\underline{\alpha}(\omega^*)$ for a set of frequencies $\omega^* \geq 0$ according to theorem 2 and plot $\underline{\alpha}(\omega^*)$ against ω^*.

3.2.2 Triangular membership functions

The obtained result requires finding roots of scalar nonlinear functions. Nevertheless, it greatly simplifies for the parameters described by fuzzy numbers with triangular membership functions.

Let us consider polynomial (3.1) with coefficients being linear affine functions of the parameters $\tilde{q}_k$ (3.2). The parameters $\tilde{q}_k$, $k = 1, \ldots, l$ are supposed to be described by fuzzy numbers with nonsymmetric triangular membership functions with core$\{\tilde{q}_k\}$ =

$q_k^0, \mathrm{supp}\{\tilde{q}_k\} = [q_k^-, q_k^+]$. The α-cut representation of $\tilde{q}_k$, $[\tilde{q}_k]_\alpha = [q_k^-(\alpha), q_k^+(\alpha)]$, is defined for $\alpha \in [0, 1]$ by linear functions

$$\begin{aligned} q_k^-(\alpha) &= (q_k^0 - q_k^-)\alpha + q_k^- , \\ q_k^+(\alpha) &= (q_k^0 - q_k^+)\alpha + q_k^+ . \end{aligned} \tag{3.32}$$

Substituting (3.32) into (3.7) one obtains

$$\begin{aligned} r_k^-(\alpha) &= (1-\alpha)r_k^-, r_k^+(\alpha) = (1-\alpha)r_k^+, \qquad (3.33) \\ r_k^- &= q_k^- - q_k^0,\ r_k^+ = q_k^+ - q_k^0,\ k = 1, \ldots, l. \end{aligned}$$

The α-cut representation of polynomial (3.1) yields

$$[\tilde{D}(s)]_\alpha = D(s, \alpha) = A(s) + (1-\alpha) \sum_{k=1}^{l} r_k B_k(s),\ r_k \in [r_k^-, r_k^+] \tag{3.34}$$

where $A(s)$ and $B_k(s)$, $k = 1, \ldots, l$ are given by (3.7). The value set of polynomial (3.34) in some point $s = \mathrm{j}\omega^*$ can be written as

$$D(\mathrm{j}\omega^*, \alpha) = A + (1-\alpha) \sum_{k=1}^{l} r_k B_k,\ r_k \in [r_k^-, r_k^+] \tag{3.35}$$

where

$$\begin{aligned} A &= A(\mathrm{j}\omega^*) = |A|e^{\mathrm{j}\theta} , \\ B_k &= B_k(\mathrm{j}\omega^*) = |B_k|e^{\mathrm{j}\phi_k},\ k = 1, \ldots, l. \end{aligned}$$

Theorem 3.2: *The maximum value of α, $0 \le \alpha \le 1$, denoted as $\underline{\alpha}(\omega^*)$ guaranteeing that $0 \in D(\mathrm{j}\omega^*, \alpha)$ for some ω^*, $0 \le \omega^* < \infty$, is*

$$\begin{aligned} \underline{\alpha}(\omega^*) &= \max \left\{ 1 - \max_{1 \le k \le l} \frac{|A||\sin(\theta - \phi_k)|}{\sum_{i=1}^{l} |r_i^*||B_i||\sin(\phi_i - \phi_k)|}, 0 \right\}, \\ &\quad \text{if } \sin(\phi_i - \phi_k) \neq 0 \text{ for some } i, k \end{aligned} \tag{3.36}$$

where

$$r_i^* = \begin{cases} r_i^-, & \text{if sign}\,(\sin(\phi_i - \phi_k)) = \text{sign}\,(\sin(\theta - \phi_k)) \\ r_i^+, & \text{if sign}\,(\sin(\phi_i - \phi_k)) \neq \text{sign}\,(\sin(\theta - \phi_k)) \end{cases}, \tag{3.37}$$

$$\begin{aligned}\underline{\alpha}(\omega^*) &= \max\left\{1-\frac{|A|}{\sum_{i=1}^{l}|r_i^*||B_i|},0\right\}, \\ &\quad \text{if } \sin(\phi_i-\phi_k)=0 \text{ and } \sin(\theta-\phi_k)=0\ \forall i,k\end{aligned} \tag{3.38}$$

where

$$r_i^* = \begin{cases} r_i^-, & \text{if } \theta=\phi_k \\ r_i^+, & \text{if } \theta=-\phi_k \end{cases}$$

and

$$\begin{aligned}\underline{\alpha}(\omega^*) &= 0, \\ &\quad \text{if } \sin(\phi_i-\phi_k)=0 \text{ and } \sin(\theta-\phi_k)\neq 0\ \forall i,k.\end{aligned}$$

Proof: We will use the result of theorem 3.1. Substituting (3.33) into (3.14) one obtains

$$\underline{\alpha}(\omega^*) = \begin{cases} \min_{1\le k\le l}\{\alpha_k\ :\ 0\le\alpha_k\le 1, \\ \qquad |A||\sin(\theta-\phi_k)| = (1-\alpha_k)\sum_{i=1}^{l}|r_i^*||B_i||\sin(\phi_i-\phi_k)|\} \\ 0, \quad \text{if } \alpha_k \text{ does not exist } \forall k=1,\ldots,l \end{cases}$$

where r_i^*, $i=1,\ldots,l$ are given by (3.37). We can write

$$\begin{aligned}\underline{\alpha}(\omega^*) &= \min_{1\le k\le l}\left\{\max\left\{1-\frac{|A||\sin(\theta-\phi_k)|}{\sum_{i=1}^{l}|r_i^*||B_i||\sin(\phi_i-\phi_k)|},0\right\}\right\} \\ &= \max\left\{1-\max_{1\le k\le l}\frac{|A||\sin(\theta-\phi_k)|}{\sum_{i=1}^{l}|r_i^*||B_i||\sin(\phi_i-\phi_k)|},0\right\}.\end{aligned}$$

Similar reasoning can be used to derive relation (3.38) from (3.15). □

Remark 3.1 The value α_∞ corresponding to degree drop of polynomial (3.34) can be determined as

$$\alpha_\infty = \max\left\{1+\frac{d_n^0}{\sum_{k=1}^{l} r_k^*\gamma_{nk}},0\right\} \tag{3.39}$$

where

$$r_k^* = \begin{cases} r_k^-, & \text{if } \operatorname{sign}(\gamma_{nk})=\operatorname{sign}(d_n^0) \\ r_k^+, & \text{if } \operatorname{sign}(\gamma_{nk})\neq\operatorname{sign}(d_n^0),\ k=1,\ldots,l \end{cases}$$

that follows immediately by substituting (3.33) into (3.12).

Remark 3.2 In the case of triangular membership functions problem of determination of $\underline{\alpha}(\omega^*)$ for some $0 \le \omega^* < \infty$ can be converted to a task of linear programming. Let us recall that

$$\underline{\alpha}(\omega^*) = \max\left\{0 \le \alpha \le 1 : D(\mathrm{j}\omega^*, \alpha) = 0\right\}.$$

If we separate the equation $D(\mathrm{j}\omega^*, \alpha) = 0$ into real and imaginary part and denote

$$\beta = 1 - \alpha,$$
$$\Phi = \begin{bmatrix} |B_1|\cos\phi_1 & \cdots & |B_l|\cos\phi_l \\ |B_1|\sin\phi_1 & \cdots & |B_l|\sin\phi_l \end{bmatrix}, \mathbf{r} = \begin{bmatrix} r_1 & \cdots & r_l \end{bmatrix}^{\mathrm{T}}, \mathbf{z} = \begin{bmatrix} -|A|\cos\theta \\ -|A|\sin\theta \end{bmatrix}$$

then we face the following standard linear programming problem:

$$\text{Minimize } \beta$$

subject to the constraints

$$\Phi\mathbf{r} = \mathbf{z}$$
$$\beta \ge 0, \quad \beta r_k^- \le r_k \le \beta r_k^+, \qquad k = 1, \ldots, l. \tag{3.40}$$

Then $\underline{\alpha}(\omega) = \max\{1 - \beta_{\min}, 0\}$.

Nevertheless, even if complexity of both algorithms solving linear programming problems (if both dimension and number of constraints are allowed to grow) and the proposed one is $\mathcal{O}(l^2)$ [30] the experiments reveal that the efficiency of the presented algorithm is much higher.

3.3 Linear interval systems with fuzzy parametric uncertainty

The result obtained in theorem 3.1 can be applied to a special form of polynomials with linear dependency of its coefficients on the parameters characterized by fuzzy numbers.

Let us consider a polynomial

$$\begin{aligned} \tilde{\Delta}(s) &= F_1(s)\tilde{P}_1(s) + \cdots + F_m(s)\tilde{P}_m(s), \\ F_i(s) &= f_{i0} + f_{i1}s + \cdots, \\ \tilde{P}_i(s) &= \tilde{p}_{i0} + \tilde{p}_{i1}s + \cdots \end{aligned} \tag{3.41}$$

where $F_i(s), i = 1, \ldots, m$ are fixed polynomials and the coefficients $\tilde{p}_{ik}$ of $\tilde{P}_i(s)$ are described by fuzzy numbers characterized by arbitrary convex, normal, continuous membership functions $\alpha = \mu_{\tilde{p}_{ik}}(p_{ik})$ sharing common confidence level α with one-element cores core$\{\tilde{p}_{ik}\} = p_{ik}^0$ and bounded supports supp$\{\tilde{p}_{ik}\} = [p_{ik}^-, p_{ik}^+]$. The α-cut representation of parameters $\tilde{p}_{ik}$, $[\tilde{p_{ik}}]_\alpha = [p_{ik}^-(\alpha), p_{ik}^+(\alpha)]$ is determined by monotonic increasing and monotonic decreasing functions

$$\begin{aligned} p_{ik}^-(\alpha) &= \mu_{\tilde{p}_{ik}}^{-1}(\alpha) \quad \text{for} \quad p_{ik}^- \le p_{ik}^-(\alpha) \le p_{ik}^0 \\ p_{ik}^+(\alpha) &= \mu_{\tilde{p}_{ik}}^{-1}(\alpha) \quad \text{for} \quad p_{ik}^0 \le p_{ik}^+(\alpha) \le p_{ik}^+, \end{aligned} \tag{3.42}$$

respectively, that at the same time characterize the α-cut representation of polynomials $\tilde{P}_i(s)$,$i = 1, \ldots, m$,

$$\begin{aligned} [\tilde{P}_i(s)]_\alpha &= P_i(s, \alpha) = p_{i0} + p_{i1}s + \cdots\,, \\ p_{ik} &\in [p_{ik}^-(\alpha), p_{ik}^+(\alpha)]. \end{aligned} \tag{3.43}$$

We are looking for minimum confidence level $\alpha_{\min} \in [0,1]$ such that the α-cut representation of polynomial $\tilde{\Delta}(s)$,

$$[\tilde{\Delta}(s)]_\alpha = \Delta(s, \alpha) = F_1(s)P_1(s, \alpha) + \cdots + F_m(s)P_m(s, \alpha) \tag{3.44}$$

is stable for $\alpha > \alpha_{\min}$ and unstable for $\alpha \le \alpha_{\min}$ under assumption that the nominal polynomial $\Delta^0(s)$,

$$\begin{aligned} \Delta^0(s) &= \Delta(s, 1) = F_1(s)P_1^0(s) + \cdots + F_m(s)P_m^0(s)\,, \\ P_i^0(s) &= P_i(s, 1) = p_{i0}^0 + p_{i1}^0 s + \cdots\,, \end{aligned}$$

is stable.

Denote

$$\begin{aligned} S_i^-(\omega, \alpha) &= -r_{i0}^-(\alpha) + r_{i2}^+(\alpha)\omega^2 - r_{i4}^-(\alpha)\omega^4 + \cdots\,, \\ S_i^+(\omega, \alpha) &= r_{i0}^+(\alpha) - r_{i2}^-(\alpha)\omega^2 + r_{i4}^+(\alpha)\omega^4 - \cdots\,, \\ T_i^-(\omega, \alpha) &= \omega(-r_{i1}^-(\alpha) + r_{i3}^+(\alpha)\omega^2 - r_{i5}^-(\alpha)\omega^4 + \cdots)\,, \\ T_i^+(\omega, \alpha) &= \omega(r_{i1}^+(\alpha) - r_{i3}^-(\alpha)\omega^2 + r_{i5}^+(\alpha)\omega^4 - \cdots), i = 1, \ldots, m \end{aligned}$$

where

$$\begin{aligned} r_{ik}^- &= p_{ik}^- - p_{ik}^0, \\ r_{ik}^+ &= p_{ik}^+ - p_{ik}^0. \end{aligned}$$

In order to determine the stability margin of (3.44) we will examine the value set $\Delta(\mathrm{j}\omega,\alpha)$ for $0\le\omega<\infty$, $0\le\alpha\le 1$. Since $P_i(s,\alpha), i=1,\dots,m$ are interval polynomials the value set $P_i(\mathrm{j}\omega,\alpha)=P_i^0(\mathrm{j}\omega)+Q_i(\mathrm{j}\omega,\alpha)$ is a rectangle in complex plane with

$$\begin{aligned} Q_i(\mathrm{j}\omega,\alpha) &= \big\{s_i(\omega,\alpha)+\mathrm{j}t_i(\omega,\alpha) : -S_i^-(\omega,\alpha)\le s_i(\omega,\alpha)\le S_i^+(\omega,\alpha)\ , \\ &\quad -T_i^-(\omega,\alpha)\le t_i(\omega,\alpha)\le T_i^+(\omega,\alpha)\big\}\,, i=1,\dots,m\,. \end{aligned}$$

Then the value set

$$\Delta(\mathrm{j}\omega,\alpha)=\Delta^0(\mathrm{j}\omega)+\mathcal{B}(\omega,\alpha) \tag{3.45}$$

where

$$\begin{aligned} \mathcal{B}(\omega,\alpha) &= \Bigg\{\sum_{i=1}^{m}\left(s_i(\omega,\alpha)+\mathrm{j}t_i(\omega,\alpha)\right)F_i(\mathrm{j}\omega) : \\ &\quad -S_i^-(\omega,\alpha)\le s_i(\omega,\alpha)\le S_i^+(\omega,\alpha), \\ &\quad -T_i^-(\omega,\alpha)\le t_i(\omega,\alpha)\le T_i^+(\omega,\alpha), i=1,\dots,m\Bigg\}\,. \end{aligned}$$

Now theorem 3.1 can be applied for determination of $\underline{\alpha}(\omega)$ – maximum value of α, $\alpha\in[0,1]$ such that zero is included in value set (3.45) – with

$$\begin{aligned} l &= 2m\,, \\ A &= \Delta^0(\mathrm{j}\omega)=F_1(\mathrm{j}\omega)P_1^0(\mathrm{j}\omega)+\cdots+F_m(\mathrm{j}\omega)P_m^0(\mathrm{j}\omega)\,, \\ r_i^-(\alpha) &= -S_i^-(\omega,\alpha), r_i^+(\alpha)=S_i^+(\omega,\alpha)\,, \\ r_{m+i}^-(\alpha) &= -T_i^-(\omega,\alpha), r_{m+i}^+(\alpha)=T_i^+(\omega,\alpha)\,, \\ B_i &= B_i(\mathrm{j}\omega)=F_i(\mathrm{j}\omega), B_{m+i}=B_{m+i}(\mathrm{j}\omega)=\mathrm{j}F_i(\mathrm{j}\omega),\ i=1,\dots,m\,. \end{aligned}$$

The stability margin $\alpha_{\min}$ will be determined from the frequency plot of $\underline{\alpha}(\omega)$ as

$$\alpha_{\min}=\max\left\{\sup_{\omega\ge 0}\underline{\alpha}(\omega),\alpha_\infty\right\} \tag{3.46}$$

where α_∞ is the value of α corresponding to degree drop of $\Delta(s,\alpha)$ that can be obtained as a solution of

$$\sum_{i=1}^{m}\sum_{k+l=n}r_{ik}(\alpha_\infty)|f_{il}|=-\sum_{i=1}^{m}\sum_{k+h=n}p_{ik}^0 f_{ih} \tag{3.47}$$

with

$$r_{ik}(\cdot) = \begin{cases} r_{ik}^{-}(\cdot), & \text{if } \left(\sum_{j=1}^{m} \sum_{k+h=n} p_{jk}^{0} f_{jh}\right) f_{il} \geq 0 \\ r_{ik}^{+}(\cdot), & \text{if } \left(\sum_{j=1}^{m} \sum_{k+h=n} p_{jk}^{0} f_{jh}\right) f_{il} < 0 \end{cases}$$

where n is degree of polynomial $\Delta^0(s)$ or $\alpha_\infty = 0$ if a solution of (3.47) does not exist.

Similar simplification as achieved in theorem 3.2 can be carried out for the coefficients of polynomials $\tilde{P}_i(s)$ in (3.41) described by triangular fuzzy numbers.

Example 3.1 – Fiat Dedra engine

In [31] the characteristic polynomial of the Fiat Dedra engine model was obtained as a fourth-order polynomial with seven uncertain parameters

$$p(s, \mathbf{q}) = a_0(\mathbf{q}) + a_1(\mathbf{q})s + a_2(\mathbf{q})s^2 + a_3(\mathbf{q})s^3 + s^4 \tag{3.48}$$

where

$$\begin{aligned}
a_0(\mathbf{q}) &= (k_{11}(k_{24} + 0.05) - k_{14}k_{21})q_1q_4q_7, \\
a_1(\mathbf{q}) &= (k_{11} - k_{14}k_{23} + k_{13}(k_{24} + 0.05))q_1q_4q_7 \\
&\quad +(k_{12}(k_{24} + 0.05) - k_{14}k_{22})q_1q_5q_7 \\
&\quad +(k_{12}k_{21} - k_{11}k_{22})q_1q_6q_7 + (k_{24} + 0.05)q_2q_5q_7 \\
&\quad +k_{21}q_2q_6q_7 + (k_{24} + 0.05)q_3q_4q_7, \\
a_2(\mathbf{q}) &= k_{13}q_1q_4q_7 + k_{12}q_1q_5q_7 + (k_{12}k_{23} - k_{13}k_{22})q_1q_6q_7 \\
&\quad +q_2q_5q_7 + k_{23}q_2q_6q_7 + q_3q_4q_7 \\
&\quad +(k_{24} + 0.05)q_5q_7 - k_{22}q_3q_6q_7 + k_{21}q_6q_7 + \\
&\quad (k_{24} + 0.05)q_2 + ((k_{24} + 0.05)k_{12} - k_{22}k_{14})q_1, \\
a_3(\mathbf{q}) &= k_{12}q_1 + q_2 + k_{23}q_6q_7 + q_5q_7 + k_{24} + 0.05.
\end{aligned}$$

k_{ij} denotes the elements of the controller gain matrix [2]

$$K = \begin{bmatrix} 0.0081 & 0.1586 & 0.8072 & -0.1202 \\ 0.0187 & 0.0848 & 0.1826 & -0.0224 \end{bmatrix}. \tag{3.49}$$

The coefficients of the characteristic polynomial (3.48) depend multilinearly on the uncertain parameters defined as a box

$$Q = \left\{ \mathbf{q} \in \Re^7 : q_i^- \leq q_i \leq q_i^+, i = 1, \ldots, 7 \right\} \tag{3.50}$$

where the vector of lower and upper bounds is given as

$$
\begin{aligned}
\mathbf{q}^- &= [q_i^-, i = 1, \ldots, 7] \\
&= [0.1261 \;\; -0.2073 \;\; 0.0357 \;\; 0.2539 \;\; 0.0100 \;\; 2.0247 \;\; 0.1000], \\
\mathbf{q}^+ &= [q_i^+, i = 1, \ldots, 7] \\
&= [3.4329 \;\; 0.1627 \;\; 0.1139 \;\; 0.5607 \;\; 0.0208 \;\; 4.4962 \;\; 1.0000],
\end{aligned}
$$

respectively.

In [2], [32] the nominal parameter values corresponding to most common operating point representing slightly loaded engine at idle speed are considered as

$$
\begin{aligned}
q_1^0 &= q_1^+ = 3.4329\,, q_2^0 = q_2^+ = 0.1627\,, q_3^0 = q_3^+ = 0.1139\,, \\
q_4^0 &= q_4^- = 0.2539\,, q_5^0 = q_5^+ = 0.0208\,, q_6^0 = q_6^- = 2.0247\,, \\
q_7^0 &= q_7^+ = 1.0000\,.
\end{aligned}
\tag{3.51}
$$

It should be noted that the nominal parameter values do not lie in the middle of the admissible intervals. The question is how far we can get away from the nominal point to preserve stability of (3.48).

The characteristic polynomial (3.48) has multilinear uncertainty structure. In [32] affine linearization is carried out – by fixing some parameters the original polynomial is changed to an affine linear interval polynomial. Such transformation leads to a necessary stability condition only, however, this information can still provide very useful insight to the original problem using effective methods.

In particular, an inspection of the coefficients of the characteristic polynomial (3.48) reveals that if q_4, q_5, q_6 and q_7 are fixed then the coefficients depend affine linearly on q_1, q_2 and q_3. In [32] the parameters q_4, q_5, q_6 and q_7 are fixed at their nominal values but for the parameters q_1, q_2 and q_3 the midpoints of the admissible intervals are chosen as "nominal" values although they do not correspond to the real nominal (operating) point (3.51). For example it means that the corresponding maximum admissible intervals, in which the parameters can lie to preserve stability of the characteristic polynomial, do not necessarily cover the real operating point. This is a serious disadvantage of that procedure.

In order to overcome the drawback mentioned above we will characterize the uncertain parameters q_1, q_2 and q_3 by fuzzy numbers $\tilde{q}_1, \tilde{q}_2$ and $\tilde{q}_3$ described by nonsymmetric triangular membership functions with

$$\mathrm{supp}\{\tilde{q}_i\} = [q_i^-, q_i^+]\,, \mathrm{core}\{\tilde{q}_i\} = q_i^0\,, i = 1, 2, 3\,. \tag{3.52}$$

Affine linearization of (3.48) by fixing q_4, q_5, q_6 and q_7 at the nominal point and characterization of q_1, q_2 and q_3 by $\tilde{q}_1, \tilde{q}_2$ and $\tilde{q}_3$ respectively form polynomial

$$\tilde{D}(s) = \tilde{d}_0 + \tilde{d}_1 s + \tilde{d}_2 s^2 + \tilde{d}_3 s^3 + s^4 \tag{3.53}$$

where

$$\begin{aligned}
\tilde{d}_0 &= 0.0006\tilde{q}_1\,,\\
\tilde{d}_1 &= 0.0182\tilde{q}_1 + 0.0384\tilde{q}_2 + 0.0070\tilde{q}_3\,,\\
\tilde{d}_2 &= 0.0384 + 0.1429\tilde{q}_1 + 0.4181\tilde{q}_2 + 0.0822\tilde{q}_3\,,\\
\tilde{d}_3 &= 0.4181 + 0.1586\tilde{q}_1 + \tilde{q}_2 + 0.0822\tilde{q}_3
\end{aligned}$$

with $\tilde{q}_i, i = 1, 2, 3$ described by (3.52).

According to (3.34) the α-cut representation of polynomial (3.53) yields

$$[\tilde{D}(s)]_\alpha = D(s, \alpha) = A(s) + (1-\alpha)\sum_{i=1}^{m} r_i B_i(s), r_i^- \le r_i \le r_i^+ \tag{3.54}$$

where

$$\begin{aligned}
A(s) &= s^4 + 1.1253s^3 + 0.6063s^2 + 0.0695s + 0.0022\,,\\
B_1(s) &= 0.1586s^3 + 0.1429s^2 + 0.0182s + 0.0006\,,\\
B_2(s) &= s^3 + 0.4181s^2 + 0.0384s\,,\\
B_3(s) &= 0.0822s^2 + 0.0070s\,,\\
r_1^- &= -3.1068, r_2^- = -0.3600, r_3^- = -0.0782,\\
r_1^+ &= r_2^+ = r_3^+ = 0\,.
\end{aligned}$$

Since the polynomial $A(s)$ is Hurwitz stable we can apply the result stated in theorem 3.2. As the polytope of polynomials (3.54) is of constant degree, then $\alpha_\infty = 0$. The plot of $\underline{\alpha}(\omega)$ against frequency is depicted in Fig. 3.1. From this plot $\alpha_{\min} = \max\{\sup_{\omega\ge 0}\underline{\alpha}(\omega), \alpha_\infty\} = 0.1103$. The corresponding plot of value set is shown in Fig. 3.2.

The obtained maximum admissible intervals of the parameters preserving stability, $[\tilde{q}_1]_{\alpha_{\min}} = [0.4909, 3.4329]$, $[\tilde{q}_2]_{\alpha_{\min}} = [-0.1665, 0.1627]$, $[\tilde{q}_3]_{\alpha_{\min}} = [0.0443, 0.1139]$, reflect better the operating conditions than those obtained in [32] ($q_1^* \in [0.4909, 3.0681]$, $q_2^* \in [-0.1665, 0.1219]$, $q_3^* \in [0.0443, 0.1053]$) which even do not include the nominal operating point (3.51).

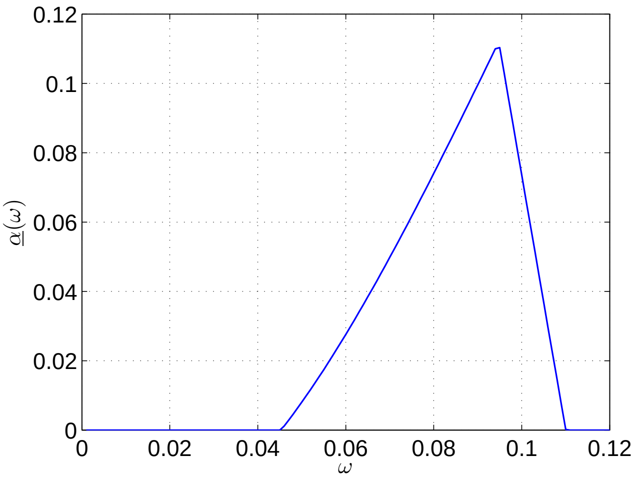

Figure 3.1: Frequency plot of $\underline{\alpha}(\omega)$ of Fiat Dedra engine

Example 3.2 – Feedback control

Let us consider a plant described by transfer function $\tilde{P}(s) = \frac{\tilde{P}_1(s)}{\tilde{P}_2(s)}$ controlled by fixed output feedback compensator $F(s) = \frac{F_1(s)}{F_2(s)}$ where

$$\begin{aligned} \tilde{P}_1(s) &= \tilde{p}_{11}s + \tilde{p}_{10}, \quad \tilde{P}_2(s) = \tilde{p}_{22}s^2 + \tilde{p}_{21}s + \tilde{p}_{20}, \\ F_1(s) &= s^2 + 2s + 2, \quad F_2(s) = s^4 + 2s^3 + 2s^2 + s \end{aligned}$$

with the parameters $\tilde{p}_{11}$, $\tilde{p}_{10}$, $\tilde{p}_{22}$, $\tilde{p}_{21}$, $\tilde{p}_{20}$ characterized by fuzzy numbers with π-shaped membership functions

$$\begin{aligned} &\mu_{\tilde{p}_{11}}(p_{11}) = \pi(p_{11}, 0.015, 0.287, 2.325),\ \mu_{\tilde{p}_{10}}(p_{10}) = \pi(p_{10}, 0.111, 0.265, 1.787), \\ &\mu_{\tilde{p}_{22}}(p_{22}) = \pi(p_{22}, 0.064, 0.215, 0.758),\ \mu_{\tilde{p}_{21}}(p_{21}) = \pi(p_{21}, 0.903, 2.060, 2.684), \\ &\mu_{\tilde{p}_{20}}(p_{20}) = \pi(p_{20}, 1.019, 2.735, 3.913). \end{aligned}$$

The π-shaped membership function $\mu(x) = \pi(x, a, b, c)$ (see Fig. 3.3) is defined as

$$\pi(x,a,b,c) = \begin{cases} 0, & \text{if } x < a \text{ or } x > c \\ 2\left(\frac{x-a}{b-a}\right)^2, & \text{if } a \le x < \frac{a+b}{2} \\ 1 - 2\left(\frac{b-x}{b-a}\right)^2, & \text{if } \frac{a+b}{2} \le x < b \\ 1 - 2\left(\frac{x-b}{b-c}\right)^2, & \text{if } b \le x < \frac{b+c}{2} \\ 2\left(\frac{c-x}{b-c}\right)^2, & \text{if } \frac{b+c}{2} \le x \le c. \end{cases}$$

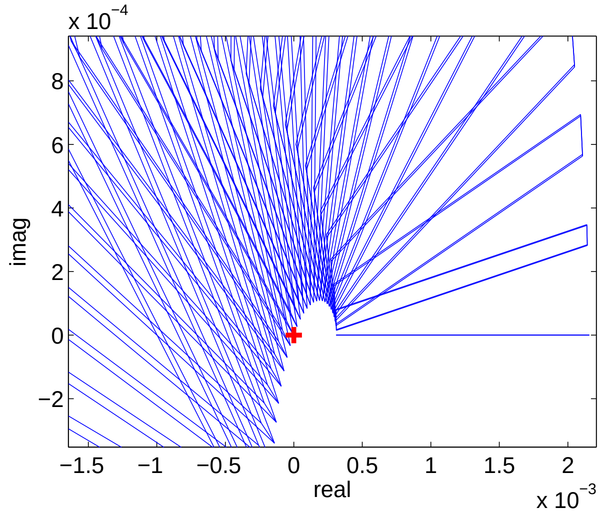

Figure 3.2: Value set plot of Fiat Dedra engine, $\alpha = 0.1103$, $\omega \in [0, 0.15]$, step 0.005

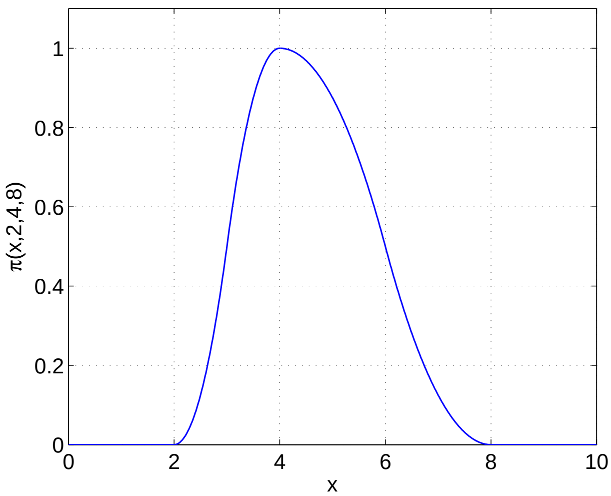

Figure 3.3: π-shaped membership function

Let us determine the minimum confidence level $\alpha_{\min} \in [0, 1]$ preserving stability of characteristic polynomial

$$\tilde{\Delta}(s) \quad = \quad F_1(s)\tilde{P}_1(s) + F_2(s)\tilde{P}_2(s)\,. \tag{3.55}$$

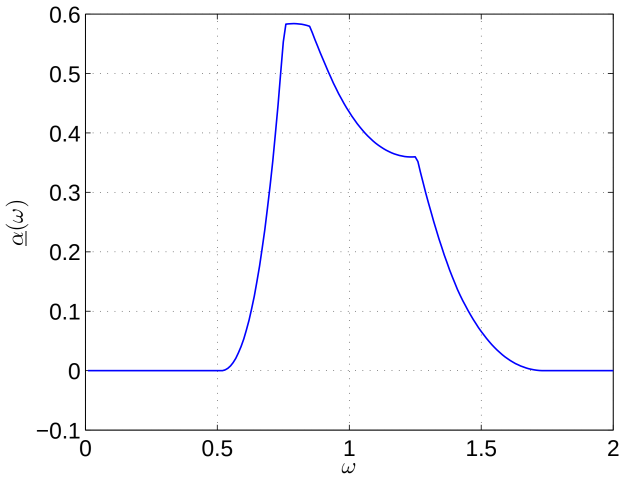

Figure 3.4: Frequency plot of $\underline{\alpha}(\omega)$ of polynomial (3.55)

Firstly let us verify that the nominal polynomial

$$\begin{aligned}\Delta^0(s) &= F_1(s)P_1^0(s) + F_2(s)P_2^0(s)\\ &= 0.215s^6 + 2.49s^5 + 7.285s^4 + 10.092s^3 + 8.369s^2 + 3.839s + 0.53\end{aligned}$$

is stable.

The corresponding frequency plot of $\underline{\alpha}(\omega)$ is shown in Fig. 3.4. From this plot we will find $\underline{\alpha}_\omega = \sup_{0\leq\omega<\infty} \underline{\alpha}(\omega) = 0.584$ and using (3.47), $\alpha_\infty = 0$. According to (3.46) the minimum confidence level $\alpha_{\min}$ preserving stability of (3.55)

$$\alpha_{\min} = \max\{\underline{\alpha}_\omega, \alpha_\infty\} = 0.584$$

and the corresponding plant parameter intervals are $[\tilde{p}_{11}]_{\alpha_{\min}} = [0.163, 1.217]$, $[\tilde{p}_{10}]_{\alpha_{\min}} = [0.195, 0.959]$, $[\tilde{p}_{22}]_{\alpha_{\min}} = [0.146, 0.463]$, $[\tilde{p}_{21}]_{\alpha_{\min}} = [1.532, 2.345]$, $[\tilde{p}_{20}]_{\alpha_{\min}} = [1.952, 3.272]$. The corresponding plot of value set is shown in Fig. 3.5.

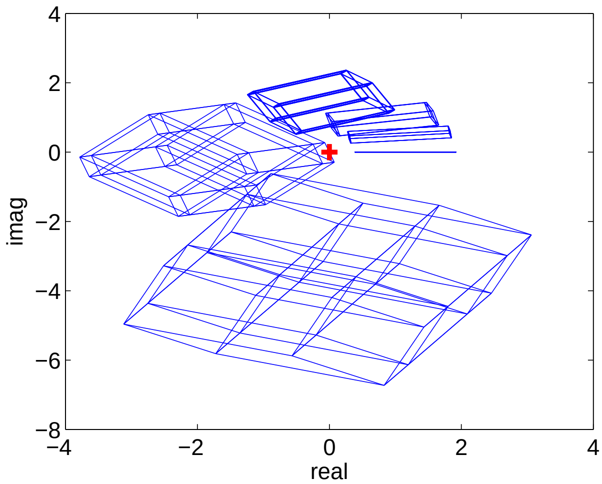

Figure 3.5: Value set plot of polynomial (3.55), $\alpha = 0.584$, $\omega \in [0, 1.1]$, step 0.2

Chapter 4

PI and PD controller with fuzzy gain and phase margin specifications

4.1 Introduction

Classical robust control approach offers several tools how to deal with uncertain systems with both structured and parametric uncertainty. The former case is usually treated within H_2 and H_∞ optimal control framework [33], [34], whereas the latter deals with interval systems with different kind of dependency of transfer function coefficients on system parameters [1], [2].

Typical task of robust control is to design a controller such that closed loop meets some requirements for any system within a prespecified set. There are various approaches solving the task. Considering linear systems the small gain theorem is applied in [35], in [36] an algebraic approach is used. Applications of robust control of nonlinear systems are described in [37] or in [38] where methods of evolutionary computation are disposed.

Disadvantage of classical approach is that uncertainty of the plant is considered the same regardless of operating conditions. Therefore the *worst-case* uncertainty occurring very rarely only in unusual conditions has to be taken into account with the same importance as *most-cases* uncertainty caused by the influence of common factors. However, in practical applications the closed loop specifications (e.g. maximum step response overshoot, maximum settling time) are typically stronger for the system that operates in typical conditions than those for the system with its parameters lying far from the normal operating point. Consequently the controller designed in such a way that it satisfies the worst-case specifications may not lead to satisfactory performance for the most-cases model. In that case, either specifications must be degraded or the family of systems must be reduced (i.e., leaving out infrequent cases, without performance guarantees for the discarded plants). Therefore a degradation of closed loop specifications towards *worst-case* uncertainty would be beneficial.

Representation of both uncertain parameters of a plant and closed loop specifications by fuzzy sets offers an elegant tool how to make it possible to consider such performance degradation. One may think of fuzzy sets as possibility distributions [39] and in such a way describe both the uncertain plant and the closed loop specifications. The models can arise from identification experiments assigning possibility distributions for the plant parameters. The core of the fuzzy plant set is supposed to contain a good approximation of the plant in majority of working conditions whereas the support is expected to cover up all the possible plants. Regarding the specifications, the core of fuzzy set characterizes most-cases closed loop performance and the support corresponds to the limit of acceptable behaviour in worst-case conditions (typically very weak requirements, e.g. stability of closed loop). The objective is to find a crisp controller such that closed loop performance of a fuzzy plant model and the controller satisfies desired specifications in terms of inclusion of respective α-cuts for all $\alpha \in [0, 1]$. A simple version of the task is accomplishing the inclusion for the cores and supports of the sets only.

There are different approaches for controller design of systems with fuzzy parametric uncertainty reported in literature. A feedforward controller based on an inverse model of such plants is addressed in [40]. In [41] design of reduced fuzzy PID controller for fuzzy interval plants using approximation of fuzzy sets by crisp intervals is discussed.

The chapter deals with linear systems with parametric uncertainty. If each coefficient of plant transfer function is described by fuzzy number the α-cuts correspond to interval systems. In [8] the closed loop specifications are defined by means of interval coefficients of characteristic polynomial. From practical point of view such formulation of fuzzy pole placement has several drawbacks. Firstly, it is not easy to specify the position of closed loop poles in order to obtain an acceptable response. More importantly, it is almost impossible to guarantee satisfactory performance by specifying admissible independent intervals for coefficients of characteristic polynomial due to complicated dependency of those coefficients on the position of poles. In [42] the fuzzy sets inclusion problem is formulated in frequency domain. A controller is to be found such that for a specified frequency range (or only set of discrete frequencies) the value set (frequency response in Nyquist diagram) of all possible closed loop plants is included in the value set of a given reference model which is supposed to be an interval system obtained from the envelope defined by the unit closed loop step responses. A control structure of two degrees of freedom with no limits on the controller order was considered. By local optimization techniques, bounds on the controller frequency response were obtained leading to determination of a single controller. The methodology was applied on PID controller design in [43].

4.2 Controller design for plants with fuzzy parametric uncertainty

The task of controller design for plants with fuzzy parametric uncertainty was defined in [8]. In controller design, the desired set of performances must be defined. As it was mentioned before in this chapter we define the controller design as an inclusion of the closed-loop image of the fuzzy plant onto the defined fuzzy performance set. If a plant $P \in \mathcal{P}$, where $\mathcal{P}$ is plant universum, is connected with a controller into the closed loop some numerical performance indexes S defined on specification space $\mathcal{S}$, $S \in \mathcal{S}$, have to be evaluated in order to assess acceptability of closed loop behaviour.

The composition of the closed-loop system jointly with the computation of performance indexes may be expressed as a map from the plant space onto the specification space (for a fixed controller), the so-called evaluation map $J_{\mathrm{C}} : \mathcal{P} \mapsto \mathcal{S}$.

For a fuzzy plant $\tilde{P}$, the evaluation map can be extended via the extension principle [39], as

$$\pi_{\tilde{J}}(z) = \max_{P \in \mathcal{P}, J_{\mathrm{C}}(P)=z} \pi_{\tilde{P}}(P) \tag{4.1}$$

giving rise to the so-called fuzzy image set $\tilde{J} := J_{\mathrm{C}}(\tilde{P})$. This set is defined on the specification space and its core determines the variety of closed-loop performances achieved by the most possible plants. The support will denote the achieved worst-case performance. Intermediate cuts define how performance is degraded.

Let us denote by P^+ and P^- the most-cases and worst-case plant, respectively, and by S^+ and S^- the desired and last acceptable specifications, respectively. The ideal case would arise if the desired specifications would be met even for worst-case plant, i.e. $J_{\mathrm{C}}(P^-) \subset S^+$. If corresponding controller cannot be found there are two commonly used options. First one consists in restriction of the set of plants to P^+ and trying to fulfil desired specifications only for most-cases plant, $J_{\mathrm{C}}(P^+) \subset S^+$. In such a case nothing is guaranteed for the plants which do not belong to the core set, even stability. Alternatively, the specification set can be relaxed to S^- and we can try to find a solution for all the plants, $J_{\mathrm{C}}(P^-) \subset S^-$. Nevertheless, behaviour of the most-cases plants may deteriorate. To avoid the loss of performance for most-cases plants and at the same time to guarantee an acceptable behaviour for all the plants it is natural to try to fulfil

$$J_{\mathrm{C}}(P^+) \subset S^+ \quad \text{AND} \quad J_{\mathrm{C}}(P^-) \subset S^- \tag{4.2}$$

at the same time.

The last approach can be generalized by defining a fuzzy target set of specifications, $\tilde{S}$, on $\mathcal{S}$, which defines the performance degradation.

The controller design problem for fuzzy plants is formulated as follows. For a given fuzzy plant $\tilde{P}$ and a fuzzy specification set $\tilde{S}$ design a fixed controller C such that $J_{\mathrm{C}}(\tilde{P}(s)) \subset \tilde{S}$. The inclusion is defined in terms of α-cuts, i.e. the controller must satisfy

$$J_{\mathrm{C}}(\tilde{P}_\alpha(s)) \subset \tilde{S}_\alpha \quad \forall \alpha \in [0, 1]. \tag{4.3}$$

In this contribution the specifications are given in terms of interval phase and gain margins and a method for PI and PD controller design for interval plants is presented. PID controllers have found a wide range of applicability and despite their simple structure have proven to be sufficient for many practical applications, including those with high performance requirements [44, 45]. Abundant amount of results has been published on tuning of PID controllers including experimental setup, pole placement, loop shaping or optimization methods [46, 47]. Stability margins are typical open loop specifications that are widely adopted for controller design [48, 49]. Their main advantage is that by adjusting only one parameter we can achieve satisfactory closed loop behaviour not only in sense of robustness but also guaranteeing performance criteria as maximum overshoot and bandwidth. Phase or gain margin specifications lying in a prescribed interval constitute a reasonable requirement for control of uncertain systems from practical point of view. Too low values cause too oscillatory transient response whilst too big values take effect in overdamped and thus very slow behaviour. Determination of set of all PI or PD controllers satisfying exact gain or phase margin leads to solving a nonlinear problem. Therefore we adopt a graphical approach displaying the controllers as curves in $k_\mathrm{P} - k_\mathrm{I}$ or $k_\mathrm{P} - k_\mathrm{D}$ plane [50], [51]. In order to guarantee gain and phase margin for an interval system we use frequency domain properties of interval plant-controller systems [1].

4.3 Problem statement

Let $\tilde{P}$ be a linear system with fuzzy parametric uncertainty described by the transfer function

$$\tilde{P}(s) = \frac{\tilde{b}_m s^m + \tilde{b}_{m-1} s^{m-1} + \cdots + \tilde{b}_0}{\tilde{a}_n s^n + \tilde{a}_{n-1} s^{n-1} + \cdots + \tilde{a}_0}\,, \quad n \geq m \tag{4.4}$$

formed by interval systems $\tilde{P}_\alpha(s)$ corresponding to α-cuts representation of $\tilde{P}(s)$,

$$\tilde{P}_\alpha(s) = \frac{[\tilde{b}_m]_\alpha s^m + [\tilde{b}_{m-1}]_\alpha s^{m-1} + \cdots + [\tilde{b}_0]_\alpha}{[\tilde{a}_n]_\alpha s^n + [\tilde{a}_{n-1}]_\alpha s^{n-1} + \cdots + [\tilde{a}_0]_\alpha}\,, \quad \tilde{P}_\alpha(s) \subset \mathcal{P}(s), \quad \alpha \in [0,1] \tag{4.5}$$

where $\tilde{b}_i, \tilde{a}_i$ denote fuzzy numbers, $[\tilde{b}_i]_\alpha, [\tilde{a}_i]_\alpha$ are their α-cuts and $\mathcal{P}(s)$ is a universal set of linear plants.

Let a fuzzy target set of specification $\tilde{S}$ defined on a universum set $\mathcal{S}$ correspond to desired phase or gain margin of closed loop characterized by membership functions μ_PM or μ_GM. In compliance with [8] let us define evaluation maps $J_\mathrm{C}^\mathrm{PM} : \mathcal{P}(s) \to \mathcal{S}$ and $J_\mathrm{C}^\mathrm{GM} : \mathcal{P}(s) \to \mathcal{S}$ that evaluate phase and gain margin of the closed loop formed by a plant $P(s) \in \mathcal{P}(s)$ and a fixed controller $C(s)$, respectively, i.e.

$$J_\mathrm{C}^\mathrm{PM}(P(s)) = \pi + \angle(C(\mathrm{j}\omega)P(\mathrm{j}\omega)) : |C(\mathrm{j}\omega)P(\mathrm{j}\omega)| = 1, \tag{4.6}$$

$$J_\mathrm{C}^\mathrm{GM}(P(s)) = |C(\mathrm{j}\omega)P(\mathrm{j}\omega)|^{-1} : \angle(C(\mathrm{j}\omega)P(\mathrm{j}\omega)) = -\pi\,. \tag{4.7}$$

The objective is to find for given $\tilde{P}(s)$ and $\tilde{S}$ a PI or PD controller, if it exists,

$$C_\mathrm{PI}(s) = k_\mathrm{P} + \frac{k_\mathrm{I}}{s}\,, \quad C_\mathrm{PD}(s) = k_\mathrm{P} + k_\mathrm{D} s \tag{4.8}$$

such that

$$J_\mathrm{C}(\tilde{P}(s)) \subset \tilde{S} \tag{4.9}$$

where $J_\mathrm{C}(\cdot) = J_\mathrm{C}^\mathrm{PM}(\cdot)$ or $J_\mathrm{C}(\cdot) = J_\mathrm{C}^\mathrm{GM}(\cdot)$.

Fuzzy set inclusion (4.9) should be understood as an interval inclusion

$$J_{\mathrm{C}}(\tilde{P}_\alpha(s)) \subset \tilde{S}_\alpha \quad \forall \alpha \in [0,1], \tag{4.10}$$

i.e.

$$J_{\mathrm{C}}(P(s)) \in \tilde{S}_\alpha \quad \forall P(s) \in \tilde{P}_\alpha(s),\ \alpha \in [0,1].$$

Let us note that the inclusion (4.10) needs to be usually satisfied for $\alpha = 0$ (the support of the plant and soft specifications corresponding to worst-case conditions) and $\alpha = 1$ (the core of the plant and hard specifications corresponding to most-cases conditions) only.

4.4 PI and PD controllers for specified gain and phase margin

In this section we will try to find all PI or PD controllers that for given fixed plant $P(s)$ exactly ensure desired phase or gain margin.

Recall that for a plant $P(s)$ and a controller $C(s)$ specification of phase margin $\phi_{\mathrm{m}} \in (0, \pi]$ implies

$$C(\mathrm{j}\omega_{\mathrm{g}})P(\mathrm{j}\omega_{\mathrm{g}}) = -\mathrm{e}^{\mathrm{j}\phi_{\mathrm{m}}} \tag{4.11}$$

for some $\omega_{\mathrm{g}} \geq 0$ and specification of gain margin $A_{\mathrm{m}} \in (1, \infty]$ implies

$$C(\mathrm{j}\omega_{\mathrm{p}})P(\mathrm{j}\omega_{\mathrm{p}}) = -\frac{1}{A_{\mathrm{m}}} \tag{4.12}$$

for some $\omega_{\mathrm{p}} \geq 0$ where ω_{g} and ω_{p} are called gain and phase crossover frequencies, respectively .

4.4.1 PI controller with phase margin specification

Let us discuss closed loop with PI controller and phase margin specification. Substituting

$$C_{\mathrm{PI}}(\mathrm{j}\omega_{\mathrm{g}}) = k_{\mathrm{P}} - \mathrm{j}\frac{k_{\mathrm{I}}}{\omega_{\mathrm{g}}} \tag{4.13}$$

into (4.11) and separating real and imaginary parts gives

$$\begin{aligned} k_{\mathrm{P}} &= \operatorname{Re}\left[\frac{-\mathrm{e}^{\mathrm{j}\phi_{\mathrm{m}}}}{P(\mathrm{j}\omega_{\mathrm{g}})}\right], \\ k_{\mathrm{I}} &= -\omega_{\mathrm{g}} \operatorname{Im}\left[\frac{-\mathrm{e}^{\mathrm{j}\phi_{\mathrm{m}}}}{P(\mathrm{j}\omega_{\mathrm{g}})}\right]. \end{aligned} \tag{4.14}$$

As a PI controller with non-negative parameters contributes a phase shift from 0 to $-\pi/2$ equation (4.14) has solution only for

$$-\pi + \phi_{\mathrm{m}} \leq \angle P(\mathrm{j}\omega_{\mathrm{g}}) < -\frac{\pi}{2} + \phi_{\mathrm{m}}$$

that corresponds to a range of frequencies $\omega_{\mathrm{g\,min}} \leq \omega_{\mathrm{g}} \leq \omega_{\mathrm{g\,max}}$,

$$\begin{aligned} \omega_{\mathrm{g\,min}} &= \begin{cases} \arg P(\mathrm{j}\omega) : \angle P(\mathrm{j}\omega) = -\frac{\pi}{2} + \phi_{\mathrm{m}}, & \text{if } \phi_{\mathrm{m}} \in (0, \frac{\pi}{2}) \\ 0, & \text{if } \phi_{\mathrm{m}} \in [\frac{\pi}{2}, \pi] \end{cases}, \\ \omega_{\mathrm{g\,max}} &= \arg P(\mathrm{j}\omega) : \angle P(\mathrm{j}\omega) = -\pi + \phi_{\mathrm{m}}. \end{aligned} \tag{4.15}$$

Substituting (4.15) into (4.14) the controllers corresponding to the frequency range endpoints become purely proportional or integral:

$$\begin{aligned} C_{\mathrm{PI_{min}}}(s) &= \begin{cases} \left(\frac{\omega_{\mathrm{g\,min}}}{|P(\mathrm{j}\omega_{\mathrm{g\,min}})|}\right) \Big/ s, & \text{if } \phi_{\mathrm{m}} \in [0, \frac{\pi}{2}] \\ -\frac{\cos(\phi_{\mathrm{m}})}{P(0)}, & \text{if } \phi_{\mathrm{m}} \in [\frac{\pi}{2}, \pi] \end{cases}, \\ C_{\mathrm{PI_{max}}}(s) &= \frac{1}{|P(\mathrm{j}\omega_{\mathrm{g\,max}})|}. \end{aligned}$$

Let us introduce

$$\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \Phi_{\mathrm{m}}) = \left\{[k_{\mathrm{P}}, k_{\mathrm{I}}] : k_{\mathrm{P}}, k_{\mathrm{I}} \geq 0, C(s) = k_{\mathrm{P}} + \frac{k_{\mathrm{I}}}{s}, J_{\mathrm{C}}^{\mathrm{PM}}(P(s)) \in \Phi_{\mathrm{m}}\right\} \tag{4.16}$$

where Φ_{m} is a set of phase margin specifications and denote the corresponding set of PI controllers

$$\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \Phi_{\mathrm{m}})(s) = \left\{C_{\mathrm{PI}}(s) : [k_{\mathrm{P}}, k_{\mathrm{I}}] \in \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \Phi_{\mathrm{m}})\right\}. \tag{4.17}$$

The points in $k_{\mathrm{P}}-k_{\mathrm{I}}$ plane lying on the frequency plot (4.14) depicted for $\omega_{\mathrm{g\,min}} \leq \omega_{\mathrm{g}} \leq \omega_{\mathrm{g\,max}}$ and some fixed $\phi_{\mathrm{m}} = \phi_{\mathrm{m0}} \in [0, \pi]$ characterize all PI controllers that guarantee phase margin ϕ_{m0} for given plant $P(s)$, $\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \phi_{\mathrm{m0}})(s)$. As each point belongs to just one value of phase margin the plot (4.14) separates the first quadrant in two subdomains

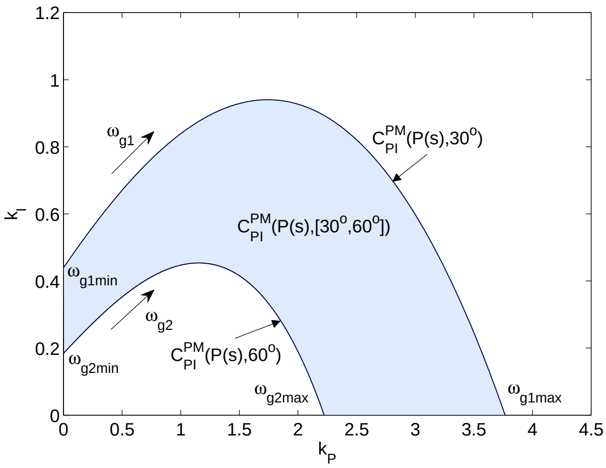

Figure 4.1: PI controllers with interval phase margin specification for plant (4.19)

– one containing the points corresponding to PI controllers with greater phase margin than ϕ_{m0}, $\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \phi_{\mathrm{m}} > \phi_{\mathrm{m0}})$, and the second one containing those PI controllers guaranteeing lower phase margin than ϕ_{m0}, $\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \phi_{\mathrm{m}} < \phi_{\mathrm{m0}})$. Therefore the set of all PI controllers guaranteeing for fixed plant $P(s)$ phase margin within an interval $[\phi_{\mathrm{m1}}, \phi_{\mathrm{m2}}]$,

$$\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), [\phi_{\mathrm{m1}}, \phi_{\mathrm{m2}}]) = \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \phi_{\mathrm{m}} \geq \phi_{\mathrm{m1}}) \cap \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \phi_{\mathrm{m}} \leq \phi_{\mathrm{m2}}) \tag{4.18}$$

corresponds to the set of points in $k_{\mathrm{P}} - k_{\mathrm{I}}$ plane delimited by frequency plots (4.14) for $\phi_{\mathrm{m}} = \phi_{\mathrm{m1}}$ and $\phi_{\mathrm{m}} = \phi_{\mathrm{m2}}$ and the lines $k_{\mathrm{P}} = 0$ and $k_{\mathrm{I}} = 0$. In Fig. 4.1 the set (4.18) is depicted for $\phi_{\mathrm{m1}} = 30°$, $\phi_{\mathrm{m2}} = 60°$ and

$$P(s) = \frac{1}{(s+1)^3}\ . \tag{4.19}$$

4.4.2 PI controller with gain margin specification

Similar procedure can be used for PI controllers and gain margin specification. Substituting (4.13) into (4.12) and treating real and imaginary part separately yields

$$\begin{aligned} k_{\mathrm{P}} &= \mathrm{Re}\left[\frac{-1}{A_{\mathrm{m}} P(\mathrm{j}\omega_{\mathrm{p}})}\right], \\ k_{\mathrm{I}} &= -\omega_{\mathrm{p}}\, \mathrm{Im}\left[\frac{-1}{A_{\mathrm{m}} P(\mathrm{j}\omega_{\mathrm{p}})}\right]. \end{aligned} \tag{4.20}$$

Since for non-negative parameters of PI controller equation (4.20) has solution only for

$$-\pi \le \angle P(\mathrm{j}\omega_{\mathrm{p}}) < -\frac{\pi}{2}$$

the frequency range is limited to $\omega_{\mathrm{p\,min}} \le \omega_{\mathrm{p}} \le \omega_{\mathrm{p\,max}}$ with

$$\begin{aligned} \omega_{\mathrm{p\,min}} &= \arg P(\mathrm{j}\omega) : \angle P(\mathrm{j}\omega) = -\frac{\pi}{2} \\ \omega_{\mathrm{p\,max}} &= \arg P(\mathrm{j}\omega) : \angle P(\mathrm{j}\omega) = -\pi \,. \end{aligned} \tag{4.21}$$

For the boundary points PI controller simplifies to

$$\begin{aligned} C_{\mathrm{PI\,min}}(s) &= \left(\frac{\omega_{\mathrm{p\,min}}}{A_{\mathrm{m}}|P(\mathrm{j}\omega_{\mathrm{p\,min}})|} \right) \Big/ s, \\ C_{\mathrm{PI\,max}}(s) &= \frac{1}{A_{\mathrm{m}}|P(\mathrm{j}\omega_{\mathrm{p\,max}})|} \,. \end{aligned}$$

Analogically to (4.16) and (4.17) let us introduce

$$\mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(P(s), \mathbf{A}_{\mathrm{m}}) = \left\{ [k_{\mathrm{P}}, k_{\mathrm{I}}] : k_{\mathrm{P}}, k_{\mathrm{I}} \ge 0, C(s) = k_{\mathrm{P}} + \frac{k_{\mathrm{I}}}{s}, J_{\mathrm{C}}^{\mathrm{GM}}(P(s)) \in \mathbf{A}_{\mathrm{m}} \right\} \tag{4.22}$$

where $\mathbf{A}_{\mathrm{m}}$ is a set of gain margin specifications and denote the corresponding set of PI controllers

$$\mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(P(s), \mathbf{A}_{\mathrm{m}})(s) = \left\{ C_{\mathrm{PI}}(s) : [k_{\mathrm{P}}, k_{\mathrm{I}}] \in \mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(P(s), \mathbf{A}_{\mathrm{m}}) \right\}. \tag{4.23}$$

Similarly to the case of phase margin, frequency plot (4.20) depicted for $\omega_{\mathrm{p\,min}} \le \omega_{\mathrm{p}} \le \omega_{\mathrm{p\,max}}$ and a fixed $A_{\mathrm{m}} = A_{\mathrm{m0}} \in (1, \infty]$ in $k_{\mathrm{P}} - k_{\mathrm{I}}$ plane characterizes all PI controllers $\mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(P(s), A_{\mathrm{m0}})(s)$ that guarantee gain margin A_{m0} for given plant $P(s)$.

The set of points in $k_{\mathrm{P}} - k_{\mathrm{I}}$ plane delimited by frequency plots (4.20) for $A_{\mathrm{m}} = A_{\mathrm{m1}}$ and $A_{\mathrm{m}} = A_{\mathrm{m2}}$ depicted for $\omega_{\mathrm{p\,min}} \le \omega_{\mathrm{p}} \le \omega_{\mathrm{p\,max}}$ and the lines $k_{\mathrm{P}} = 0$ and $k_{\mathrm{I}} = 0$,

$$\mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(P(s), [A_{\mathrm{m1}}, A_{\mathrm{m2}}]) = \mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(P(s), A_{\mathrm{m}} \ge A_{\mathrm{m1}}) \cap \mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(P(s), A_{\mathrm{m}} \le A_{\mathrm{m2}}), \tag{4.24}$$

characterizes all PI controllers guaranteeing interval gain margin $[A_{\mathrm{m1}}, A_{\mathrm{m2}}]$ for that plant $P(s)$, see Fig. 4.2 for the plant (4.19) and $A_{\mathrm{m1}} = 2$, $A_{\mathrm{m2}} = 3$.

For PD controller with transfer function $C_{\mathrm{PD}}(s) = k_{\mathrm{P}} + k_{\mathrm{D}}(s)$ similar results as those obtained above for PI controller can be accomplished.

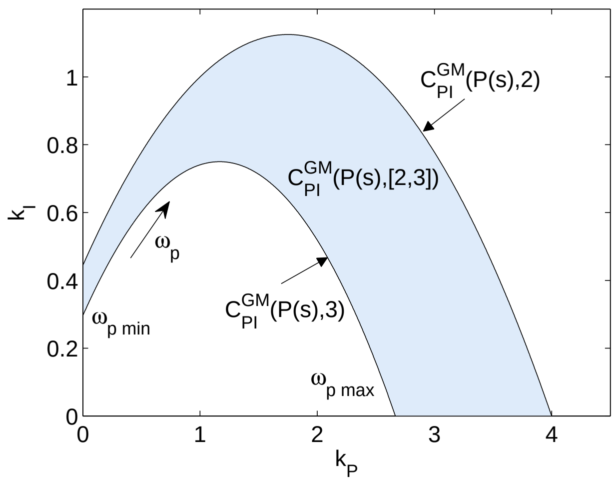

Figure 4.2: PI controllers with interval gain margin specification for plant (4.19)

4.5 Interval Control Systems

In this section we recall useful frequency domain properties of systems with uncertain parameters. Let us consider unity feedback system with

$$C(s) = \frac{c_1(s)}{c_2(s)} \ , \qquad P(s) = \frac{b(s)}{a(s)} \ . \tag{4.25}$$

We suppose that controller $C(s)$ is fixed but plant $P(s)$ contains uncertain parameters that appear as interval coefficients of polynomials $b(s)$ and $a(s)$, i.e.

$$b(s) = b_0 + b_1 s + \cdots + b_m s^m, \tag{4.26}$$

$$a(s) = a_0 + a_1 s + \cdots + a_n s^n \tag{4.27}$$

where $b_k \in [b_k^-, b_k^+]$, $k = 1, \ldots, m$ and $a_k \in [a_k^-, a_k^+]$, $k = 1, \ldots, n$. Let us define two set of polynomials (called interval polynomials)

$$\mathbf{b}(s) = \{b(s) = b_0 + b_1 s + \cdots + b_m s^m : b_k \in [b_k^-, b_k^+],\ k = 1, \ldots, m\} \tag{4.28}$$

$$\mathbf{a}(s) = \{a(s) = a_0 + a_1 s + \cdots + a_n s^n : a_k \in [a_k^-, a_k^+],\ k = 1, \ldots, n\} \tag{4.29}$$

that form a set of linear systems (referred to as an interval system)

$$\mathbf{P}(s) = \left\{ \frac{b(s)}{a(s)} : b(s) \in \mathbf{b}(s), a(s) \in \mathbf{a}(s) \right\} \tag{4.30}$$

that is for sake of simplicity written as

$$\mathbf{P}(s) = \frac{\mathbf{b}(s)}{\mathbf{a}(s)} \ . \tag{4.31}$$

Let us assign four Kharitonov polynomials to both interval polynomials $\mathbf{b}(s)$ and $\mathbf{a}(s)$:

$$\begin{aligned}
K^1_{\mathbf{p}}(s) &= p_0^- + p_1^- s + p_2^+ s^2 + p_3^+ s^3 + p_4^- s^4 + p_5^- s^5 + \cdots ,\\
K^2_{\mathbf{p}}(s) &= p_0^- + p_1^+ s + p_2^+ s^2 + p_3^- s^3 + p_4^- s^4 + p_5^+ s^5 + \cdots ,\\
K^3_{\mathbf{p}}(s) &= p_0^+ + p_1^- s + p_2^- s^2 + p_3^+ s^3 + p_4^+ s^4 + p_5^- s^5 + \cdots ,\\
K^4_{\mathbf{p}}(s) &= p_0^+ + p_1^+ s + p_2^- s^2 + p_3^- s^3 + p_4^+ s^4 + p_5^+ s^5 + \cdots
\end{aligned}$$

where $\mathbf{p} \in \{\mathbf{b}, \mathbf{a}\}$, $\{p_k^-, p_k^+\} \in \left\{ \{b_k^-, b_k^+\}, \{a_k^-, a_k^+\} \right\}$, $k = 1, \ldots, \max\{m, n\}$. Denote $\mathcal{K}_{\mathbf{b}}(s) = \{K^1_{\mathbf{b}}(s), K^2_{\mathbf{b}}(s), K^3_{\mathbf{b}}(s), K^4_{\mathbf{b}}(s)\}$ and $\mathcal{K}_{\mathbf{a}}(s) = \{K^1_{\mathbf{a}}(s), K^2_{\mathbf{a}}(s), K^3_{\mathbf{a}}(s), K^4_{\mathbf{a}}(s)\}$.

For both interval polynomials $\mathbf{b}(s)$ and $\mathbf{a}(s)$ let us introduce four Kharitonov segments joining the following pairs of Kharitonov polynomials:

$$\mathcal{S}_{\mathbf{p}}(s) = \{[K^1_{\mathbf{p}}(s), K^2_{\mathbf{p}}(s)], [K^1_{\mathbf{p}}(s), K^3_{\mathbf{p}}(s)], [K^2_{\mathbf{p}}(s), K^4_{\mathbf{p}}(s)], [K^3_{\mathbf{p}}(s), K^4_{\mathbf{p}}(s)]\}$$

where $\mathbf{p} \in \{\mathbf{b}, \mathbf{a}\}$, $\{p_k^-, p_k^+\} \in \left\{ \{b_k^-, b_k^+\}, \{a_k^-, a_k^+\} \right\}$, $k = 1, \ldots, \max\{m, n\}$. Line segment of two polynomials $\delta_1(s)$ and $\delta_2(s)$ is defined as one parameter family of polynomials

$$[\delta_1(s), \delta_2(s)] := \{\delta_\lambda(s) : \delta_\lambda(s) = \lambda\delta_1(s) + (1-\lambda)\delta_2(s),\ \lambda \in [0,1]\}.$$

Using notation (4.31) define the extremal systems as

$$\mathbf{P}_{\mathrm{E}}(s) := \frac{\mathcal{K}_{\mathbf{b}}}{\mathcal{S}_{\mathbf{a}}} \cup \frac{\mathcal{S}_{\mathbf{b}}}{\mathcal{K}_{\mathbf{a}}} \tag{4.32}$$

and Kharitonov systems as

$$\mathbf{P}_{\mathrm{K}}(s) := \frac{\mathcal{K}_{\mathbf{b}}}{\mathcal{K}_{\mathbf{a}}} \ . \tag{4.33}$$

Controller $C(s)$ is said to satisfy the vertex condition if the polynomials $c_i(s)$ can be written as

$$c_i(s) = s^{t_i}(d_i s + e_i)u_i(s)r_i(s), \qquad i = 1, 2 \tag{4.34}$$

where t_i are non-negative integers, d_i, e_i are arbitrary real numbers, $u_i(s)$ is an anti-Hurwitz polynomial and $r_i(s)$ is an even or odd polynomial.

Firstly let us recall Generalized Kharitonov Theorem – an important result that states that robust stability of closed loop over whole set $\mathbf{P}(s)$ can be reduced to testing stability over a smaller set.

Theorem 4.1 (Generalized Kharitonov Theorem [52]): *Closed loop system is stable for all $P(s) \in \mathbf{P}(s)$ if and only if it is stable for all $P(s) \in \mathbf{P}_{\mathrm{E}}(s)$. If in addition $C(s)$ satisfies the vertex condition, feedback system is stable for all $P(s) \in \mathbf{P}(s)$ if and only if it is stable for all $P(s) \in \mathbf{P}_{\mathrm{K}}(s)$.*

Next we mention a very useful result concerning the boundary of image set of interval system $\mathbf{P}(s)$ evaluated in some point $s = \mathrm{j}\omega$.

Theorem 4.2 (Boundary Generating Property [25]):

$$\partial\mathbf{P}(\mathrm{j}\omega) \in \mathbf{P}_{\mathrm{E}}(\mathrm{j}\omega) \tag{4.35}$$

where symbol ∂ stands for boundary of a set.

Both theorems provide an efficient tool for checking important properties of interval plant-controller systems. They state that stability of closed loop composed of a fixed controller and an interval plant and the boundary of image set of an interval plant can be simplified to testing those properties for 32 one parameter systems or in a special case even for 16 fixed systems only regardless of the order of the plant. We use those results for finding the set of all PI (PD) controllers guaranteeing interval phase or gain margin specification for an interval system.

4.6 Main result

Denote

$$\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(\mathbf{P}(s), \phi_{\mathrm{m}}) = \bigcup_{P(s)\in\mathbf{P}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \phi_{\mathrm{m}}) \tag{4.36}$$

and

$$\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(\mathbf{P}(s), \Phi_{\mathrm{m}}) = \bigcap_{P(s)\in\mathbf{P}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \Phi_{\mathrm{m}}) \tag{4.37}$$

where $\mathbf{P}(s)$ is an interval system (4.30), ϕ_{m} is a fixed phase margin specification, Φ_{m} is a set of phase margin specifications and $\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \phi_{\mathrm{m}})$ and $\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \Phi_{\mathrm{m}})$ are given by (4.16).

The set (4.36) is determined by frequency plots (4.14) depicted for all $P(s) \in \mathbf{P}(s)$ for frequency ranges (different for each $P(s)$) given by (4.15). Since $P(\mathrm{j}\omega)$ is connected set for any $\omega \in \Re$ the set (4.36) is also connected.

Lemma:

$$\partial \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(\mathbf{P}(s),\phi_{\mathrm{m}}) \subset \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(\mathbf{P}_{\mathrm{E}}(s),\phi_{\mathrm{m}}). \tag{4.38}$$

$\square$

Proof: Inclusion (4.38) immediately follows from Boundary Generating Property (4.35) since the complex mapping described by (4.14) is one-to-one continuous for $\phi_{\mathrm{m}} \in [0,\pi]$ and $\mathbf{P}(\mathrm{j}\omega) \neq 0$. $\square$

Now we can state the main result of the chapter.

Theorem 4.3:

$$\begin{aligned}\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(\mathbf{P}(s),[\phi_{\mathrm{m1}},\phi_{\mathrm{m2}}]) \;=\;& \bigcap_{P(s)\in\mathbf{P}_{\mathrm{E}}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s),\phi_{\mathrm{m}} \geq \phi_{\mathrm{m1}}) \;\bigcap \\ & \bigcap_{P(s)\in\mathbf{P}_{\mathrm{E}}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s),\phi_{\mathrm{m}} \leq \phi_{\mathrm{m2}}).\end{aligned} \tag{4.39}$$

Proof: Using (4.37) we have

$$\begin{aligned}\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(\mathbf{P}(s),[\phi_{\mathrm{m1}},\phi_{\mathrm{m2}}]) \;=\;& \bigcap_{P(s)\in\mathbf{P}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s),\phi_{\mathrm{m}} \geq \phi_{\mathrm{m1}}) \;\bigcap \\ & \bigcap_{P(s)\in\mathbf{P}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s),\phi_{\mathrm{m}} \leq \phi_{\mathrm{m2}}).\end{aligned} \tag{4.40}$$

We prove that $\bigcap_{P(s)\in\mathbf{P}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s),\phi_{\mathrm{m}} \geq \phi_{\mathrm{m1}}) = \bigcap_{P(s)\in\mathbf{P}_{\mathrm{E}}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s),\phi_{\mathrm{m}} \geq \phi_{\mathrm{m1}})$ by proving equality of the complements of both sets.

At first we prove that for any

$$C_{\mathrm{PI1}} \notin \bigcap_{P(s)\in\mathbf{P}_{\mathrm{E}}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s),\phi_{\mathrm{m}} \geq \phi_{\mathrm{m1}}) \quad\Longrightarrow\quad C_{\mathrm{PI1}} \notin \bigcap_{P(s)\in\mathbf{P}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s),\phi_{\mathrm{m}} \geq \phi_{\mathrm{m1}}).$$

Indeed, for any $P_1(s) \in \mathbf{P}_{\mathrm{E}}(s) : C_{\mathrm{PI1}} \notin \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P_1(s),\phi_{\mathrm{m}} \geq \phi_{\mathrm{m1}})$ then always exists $P_2(s) \in \mathbf{P}_{\mathrm{E}}(s)$ such that $C_{\mathrm{PI1}} \notin \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P_2(s),\phi_{\mathrm{m}} \geq \phi_{\mathrm{m1}})$. One can take for instance $P_2(s) = P_1(s)$.

Let us prove the opposite implication:

$$C_{\mathrm{PI1}} \notin \bigcap_{P(s)\in\mathbf{P}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s),\phi_{\mathrm{m}} \geq \phi_{\mathrm{m1}}) \Longrightarrow C_{\mathrm{PI1}} \notin \bigcap_{P(s)\in\mathbf{P}_{\mathrm{E}}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s),\phi_{\mathrm{m}} \geq \phi_{\mathrm{m1}}) \tag{4.41}$$

Let us consider $P_1(s) \in \mathbf{P}(s)$ such that the controller $C_{\mathrm{PI1}} \notin \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P_1(s),\phi_{\mathrm{m}} \geq \phi_{\mathrm{m1}})$. Then using Lemma 1

$$\begin{aligned}C_{\mathrm{PI1}} \;\notin\;& \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(\mathbf{P}(s),\phi_{\mathrm{m}} \geq \phi_{\mathrm{m1}}) \cap \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(\mathbf{P}(s),\phi_{\mathrm{m1}}) \\ & \subset \partial\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(\mathbf{P}(s),\phi_{\mathrm{m1}}) \subset \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(\mathbf{P}_{\mathrm{E}}(s),\phi_{\mathrm{m1}})\,.\end{aligned}$$

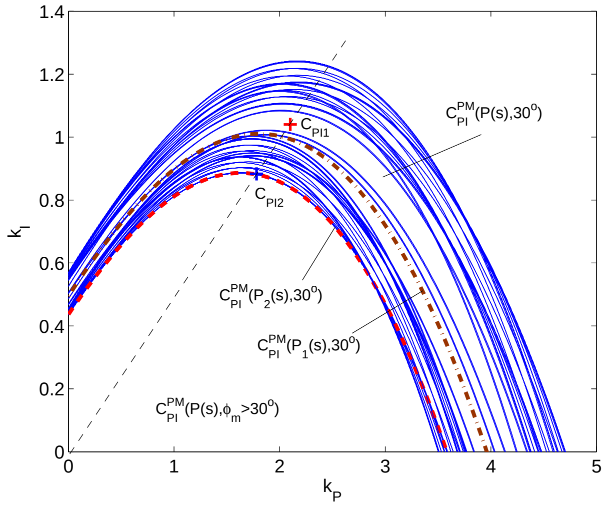

Figure 4.3: PI controller for interval system

Then always exists $P_2(s) \in \mathbf{P}_{\mathrm{E}}(s)$ such that $C_{\mathrm{PI1}} \notin \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P_2(s), \phi_{\mathrm{m}} \geq \phi_{\mathrm{m1}})$. $P_2(s)$ is chosen in such a way that the set $\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P_2(s), \phi_{\mathrm{m1}})$ contains the point $C_{\mathrm{PI2}} : C_{\mathrm{PI2}} \in \partial\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P_2(s), \phi_{\mathrm{m1}}), \angle C_{\mathrm{PI2}} = \angle C_{\mathrm{PI1}}$, see Fig. 4.3. That completes the proof of implication (4.41).

Equality $\bigcap_{P(s)\in\mathbf{P}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \phi_{\mathrm{m}} \leq \phi_{\mathrm{m2}}) = \bigcap_{P(s)\in\mathbf{P}_{\mathrm{E}}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \phi_{\mathrm{m}} \leq \phi_{\mathrm{m2}})$ can be proved in a similar way. $\square$

Theorem 4.3 suggests a simple graphical way to obtain all PI controllers that guarantee the interval phase margin specification for an interval system by plotting sets $\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \phi_{\mathrm{m1}})$ and $\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(P(s), \phi_{\mathrm{m2}})$ in $k_{\mathrm{P}} - k_{\mathrm{I}}$ plane for all $P(s) \in \mathbf{P}_{\mathrm{E}}(s)$, i.e. for 32 one parameter plants. The set $\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(\mathbf{P}(s), [30°, 60°])$ for

$$\mathbf{P}(s) = \frac{b_0}{s^3 + a_2 s^2 + a_1 s + a_0}, \qquad b_0 \in [0.8, 1.2],\ a_2 \in [2.8, 3.2],\ a_1 \in [2.9, 3.15],\ a_0 \in [0.9, 1.1] \tag{4.42}$$

is depicted in Fig. 4.4.

The result becomes even easier for the interval gain margin specification. Denote

$$\mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(\mathbf{P}(s), \mathbf{A}_{\mathrm{m}}) = \bigcap_{P(s)\in\mathbf{P}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(P(s), \mathbf{A}_{\mathrm{m}}) \tag{4.43}$$

where $\mathbf{A}_{\mathrm{m}}$ denotes a set of gain margin specifications.

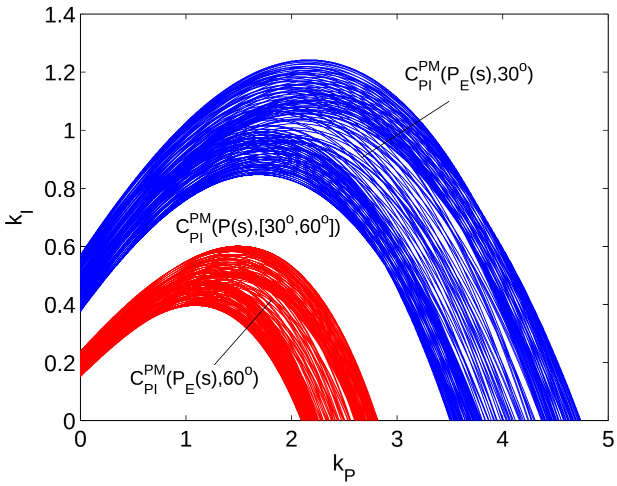

Figure 4.4: PI controllers for interval system with interval margin specification

Theorem 4.4:

$$\begin{aligned}\mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(\mathbf{P}(s),[A_{\mathrm{m1}},A_{\mathrm{m2}}]) &= \bigcap_{P(s)\in\mathbf{P}_{\mathrm{K}}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(P(s),A_{\mathrm{m}}\geq A_{\mathrm{m1}}) \bigcap \\ & \bigcap_{P(s)\in\mathbf{P}_{\mathrm{K}}(s)} \mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(P(s),A_{\mathrm{m}}\leq A_{\mathrm{m2}}). \qquad (4.44)\end{aligned}$$

The proof of this theorem follows the proof of theorem 4.3 and uses theorem 4.1 and the fact that a PI controller satisfies the vertex condition (4.34). Theorem 4.4 states that all PI controllers meeting the interval gain margin specifications for an interval system can be found by frequency plots $\mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(P(s),\phi_{\mathrm{m1}})$ and $\mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(P(s),\phi_{\mathrm{m2}})$ for 16 fixed systems only.

Using theorems 4.3 and 4.4 the set of all PI controllers satisfying (4.9) can be found as

$$\begin{aligned}\mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(s) &= \bigcap_{\alpha\in[0,1]} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(\tilde{P}_{\alpha}(s),\tilde{S}_{\alpha})(s) = \bigcap_{\alpha\in[0,1]} \mathbf{C}_{\mathrm{PI}}^{\mathrm{PM}}(\tilde{P}_{\alpha\mathrm{E}}(s),\tilde{S}_{\alpha})(s), \\ \mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(s) &= \bigcap_{\alpha\in[0,1]} \mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(\tilde{P}_{\alpha}(s),\tilde{S}_{\alpha})(s) = \bigcap_{\alpha\in[0,1]} \mathbf{C}_{\mathrm{PI}}^{\mathrm{GM}}(\tilde{P}_{\alpha\mathrm{K}}(s),\tilde{S}_{\alpha})(s) \qquad (4.45)\end{aligned}$$

where $\tilde{P}_{\alpha\mathrm{E}}(s)$ and $\tilde{P}_{\alpha\mathrm{K}}(s)$ are the extremal and Kharitonov systems associated to $\tilde{P}_{\alpha}(s)$, respectively.

Similar results can be derived for finding the set of all PD controllers guaranteeing the specified fuzzy phase or gain margin for systems with fuzzy parametric uncertainty.

Naturally, the obtained results can be combined in order to find PI or PD controllers satisfying phase and gain margin specifications at the same time.

The advantages of the presented method over the existing ones and its applicability will be demonstrated on two examples.

Example 4.1 – DC motor velocity control

At first, angular velocity control of a laboratory model of DC motor shown in Fig. 4.5 is presented. The model is equipped with a magnetic damping system that enables to change dynamic behaviour of the model. Responses of the model around an operating point on step input voltage from 0.5 to 0.7 corresponding to most-cases (core of the plant) and worst-case (support of the plant) conditions are depicted in Fig. 4.6. In practice, the core of the plant reflects damping variations caused by common changes of environmental temperature whereas the support of the plant corresponds to a damage of the bearings that occurs very rarely.

Suppose it is desired to find a PI controller such that the overshoot of transient response of the core of the plant does not exceed 10% and for the support of the plant is maximally 20%.

Identification from the step responses inspires us to consider the following model with fuzzy parameter uncertainty:

$$\tilde{P}(s) = \frac{\tilde{b}_0}{s^2 + \tilde{a}_1 s + \tilde{a}_0} \tag{4.46}$$

where the parameters are characterized by trapezoidal membership functions

$$\begin{aligned}
\tilde{b}_0 &= \text{trap}(2.349, 2.646, 2.91, 3.221), \\
\tilde{a}_1 &= \text{trap}(5.465, 6.508, 7.159, 8.617), \\
\tilde{a}_0 &= \text{trap}(2.326, 5.291, 5.82, 9.762).
\end{aligned}$$

The step responses of the core and the support of the model are depicted in Fig. 4.6.

The requirement of 10% overshoot corresponds for a second order process to phase margin around 55°, 20% overshoot approximately to 45°. Since for the second order system the values of phase margin over 75° usually lead to overdamped and thus too slow response it is reasonable to choose the set of phase margin specifications as trapezoidal

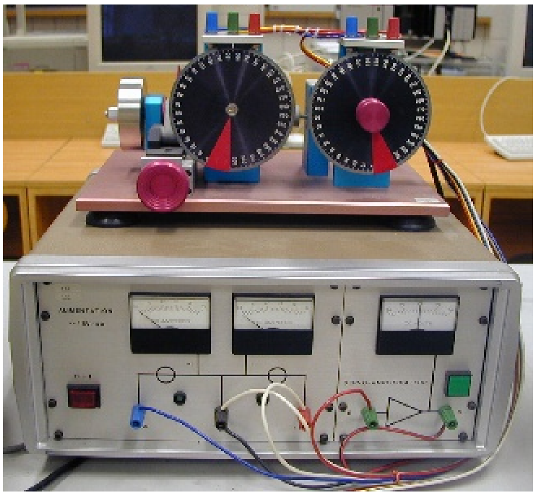

Figure 4.5: Laboratory model of DC motor

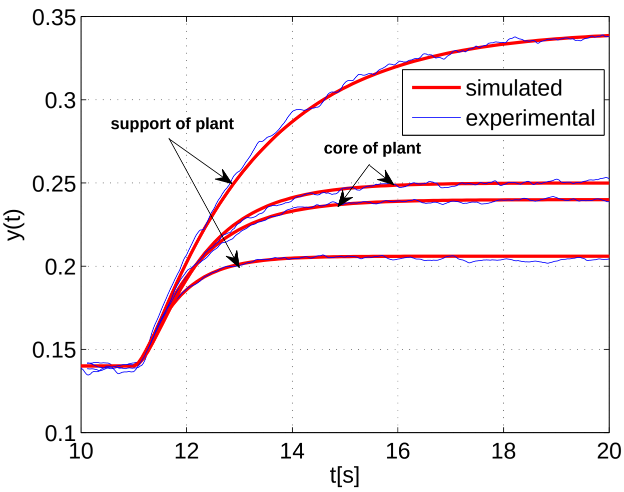

Figure 4.6: Step responses of the plant

fuzzy set $\tilde{S} = \text{trap}(45°, 52°, 60°, 75°)$. Let us find a PI controller (or all of them) that simultaneously satisfies the inclusion (4.10) for $\alpha = 0$ and $\alpha = 1$, i.e. that guarantees for any plant of the core of the plant the phase margin between 52° and 60° and for any plant of the support of the plant the phase margin between 45° and 75°.

The set of all PI controllers satisfying inclusion (4.10) for the support of the plant and specifications (i.e. for $\alpha = 0$) and for the core of the plant and specifications (i.e. for $\alpha = 1$) is shown in Fig. 4.7 and Fig. 4.8, respectively. Using (4.39) the set of all PI controllers satisfying that inclusion for all $\alpha \in [0, 1]$ is delimited by the cyan line in Fig. 4.9. From that set we choose PI controller with $k_\mathrm{P} = 6.8$, $k_\mathrm{I} = 5.8$ (denoted by $\circ$) due to plant input saturation and high integral action since the experience shows that the controllers on each boundary with maximum integral part generally produce the best values for the speed of response and performance, see [50].

The simulated and experimental closed loop step reference responses shown in Fig. 4.10 and Fig. 4.11, respectively, confirm that the desired behaviour is achieved, i.e. the overshoot for the core of the plant does not exceed 10% (red line) and for the support of the plant does not exceed 20% (blue line).

Now let us use the classical robust control approach. There are two possibilities – to design a PI controller satisfying the support specifications for the support of the plant and disregard the behaviour for the core of the plant or, vice versa, to find a controller with respect to the core of the plant regardless of the specifications for the support of the plant. Applying the former approach and using the algorithm described in [51] generalized according to [52] one can find the PI controller denoted by $\times$. Comparison of reference step responses of that controller (dash line) and the controller designed using the presented method (solid line) depicted in Fig. 4.12 shows small difference for the support of the plant (thin line) whereas for the core of the plant (thick line) that controller does not meet the required performance. Focusing only on the core of the plant the algorithm [51] leads to the PI controller denoted by $+$ that on the other hand exposes unsatisfactory behaviour for the support of the plant (see Fig. 4.13).

Example 4.2 – Control of active suspension

The second example applies the presented method on a PD controller design with respect to disturbance attenuation. Let us consider control of a quarter car active

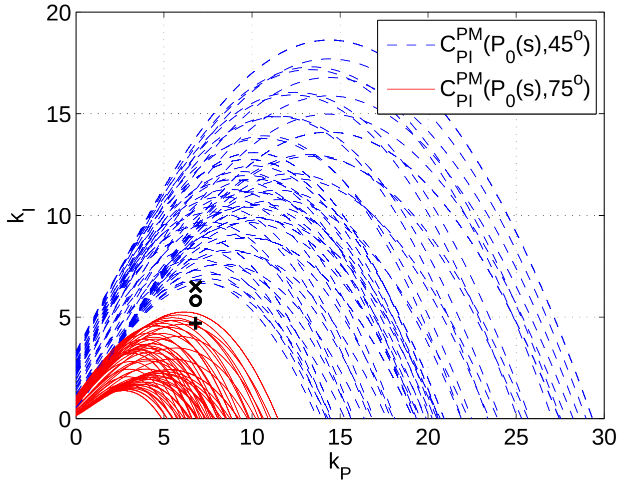

Figure 4.7: PI controllers for support of plant

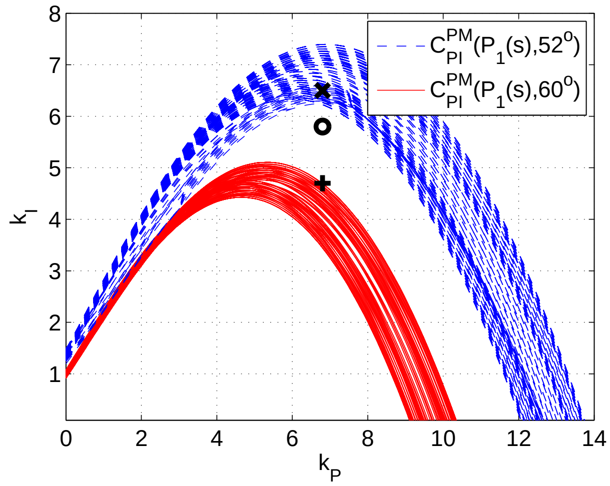

Figure 4.8: PI controllers for core of plant

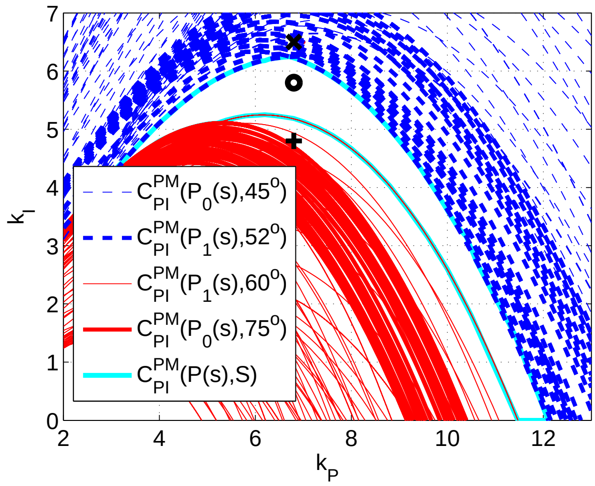

Figure 4.9: PI controllers for fuzzy linear plant with fuzzy phase margin specification

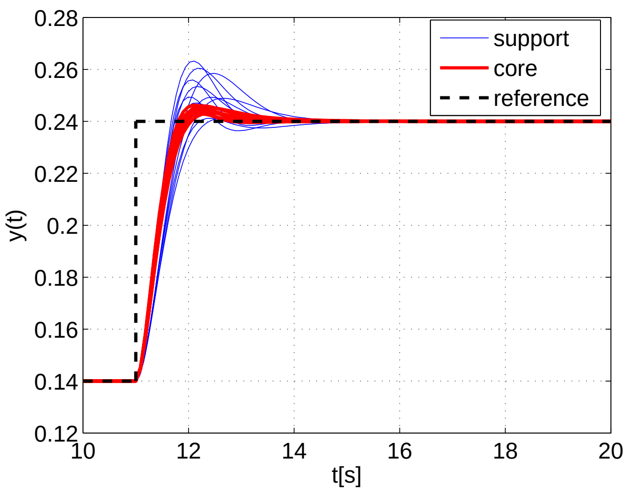

Figure 4.10: Simulation of closed loop with PI controller

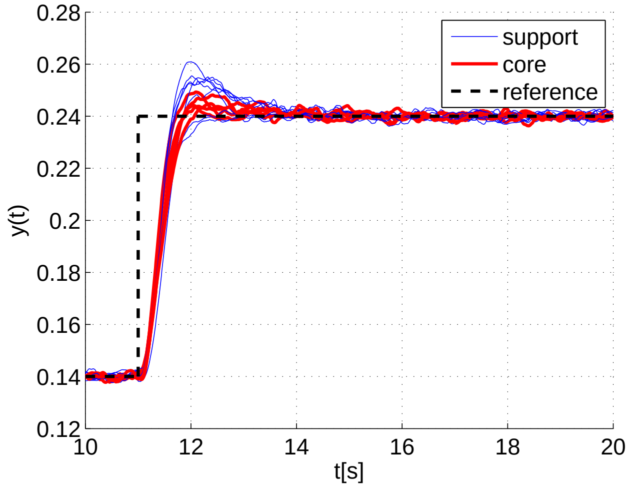

Figure 4.11: Experimental closed loop with PI controller

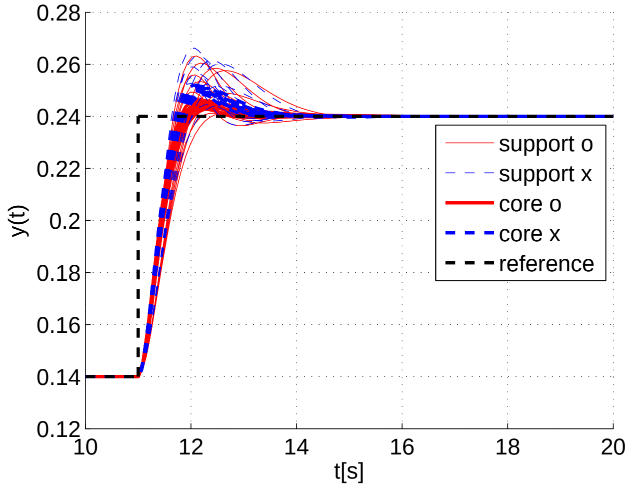

Figure 4.12: Responses of controller not satisfactory for core of plant

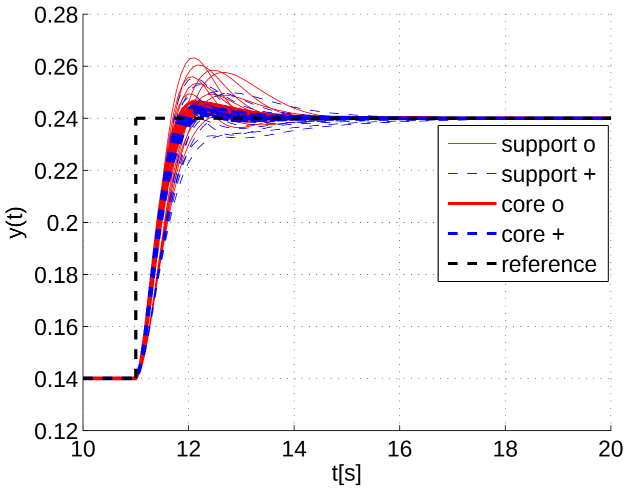

Figure 4.13: Responses of controller not satisfactory for support of plant

suspension system (Fig. 4.14). Input to the system $u(t)$ is an active force generated by a pneumatic actuator, output $y(t)$ is the displacement of car body, $z_1(t)$. The road profile $r(t)$ represents disturbance signal, m_1 and m_2, respectively, correspond to the masses of the car body and wheel, k_1 and k_2 denote the stiffness of springs representing tire compressibility and car suspension, respectively, and the friction constant b represents the shock absorber.

State-space equations of the system around an equilibrium can be written as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) + \mathbf{d}\xi(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{4.47}$$

with

$$\begin{aligned} \mathbf{x}(t) &= [\dot{z}_1(t) \quad z_1(t) \quad \dot{z}_2(t) \quad z_2(t) \quad \dot{r}(t)]^{\mathrm{T}}, \\ \mathbf{A} &= \begin{bmatrix} -\frac{b}{m_1} & -\frac{k_1}{m_1} & \frac{b}{m_1} & \frac{k_1}{m_1} & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{b}{m_2} & \frac{k_1}{m_2} & -\frac{b}{m_2} & -\frac{k_1+k_2}{m_2} & \frac{k_2}{m_2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \frac{1}{m_1} & 0 & -\frac{1}{m_2} & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \\ \mathbf{C} &= [0 \quad 1 \quad 0 \quad 0 \quad 0], \mathbf{d} = [0 \quad 0 \quad 0 \quad 0 \quad 1]^{\mathrm{T}} \end{aligned}$$

where $\xi(t)$ is a time-varying function satisfying $\dot{r}(t) = \xi(t)$.

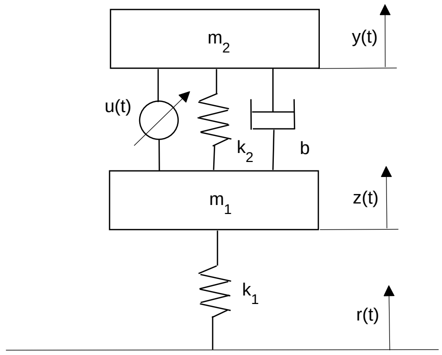

Figure 4.14: Quarter car active suspension system

Transfer function of the plant (4.47) from $u(t)$ to $y(t)$ yields

$$\begin{aligned} P(s) = \frac{Y(s)}{U(s)} &= \frac{b_2 s^2 + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \\ &= \frac{\frac{1}{m_1} s^2 + \frac{k_2}{m_1 m_2}}{s^4 + \left(\frac{b}{m_1} + \frac{b}{m_2}\right) s^3 + \left(\frac{k_1}{m_1} + \frac{k_1}{m_2} + \frac{k_2}{m_2}\right) s^2 + \frac{k_1 b}{m_1 m_2} s + \frac{k_1 k_2}{m_1 m_2}} . \end{aligned} \tag{4.48}$$

Suppose that in normal operating conditions the parameters of the plant equal to their nominal values, $k_1 = k_1^0$, $k_2 = k_2^0$, $b = b^0$, $m_1 = m_1^0$, $m_2 = m_2^0$ but occasionally they can take any fixed value within prescribed intervals, $k_1 \in [k_1^-, k_1^+]$, $k_2 \in [k_2^-, k_2^+]$, $b \in [b^-, b^+]$, $m_1 \in [m_1^-, m_1^+]$, $m_2 \in [m_2^-, m_2^+]$. Therefore it is reasonable to characterize the plant (4.48) by a linear model with fuzzy parametric uncertainty

$$\tilde{P}(s) = \frac{\tilde{b}_2 s^2 + \tilde{b}_0}{s^4 + \tilde{a}_3 s^3 + \tilde{a}_2 s^2 + \tilde{a}_1 s + \tilde{a}_0} \tag{4.49}$$

with the coefficients described by fuzzy numbers with triangular membership functions

$$\begin{aligned} b_i &= \mathrm{tri}(b_i^-, b_i^0, b_i^+), \quad i = 0, 2, \\ a_j &= \mathrm{tri}(a_j^-, a_j^0, a_j^+), \quad j = 0, \ldots, 3 \end{aligned}$$

where b_i^0 and a_j^0 are given by nominal values of the parameters and b_i^-, b_i^+, a_j^-, a_j^+ correspond to minimum and maximum values of the corresponding coefficients.

Let us consider the following numerical values: $k_1^0 = 1.3 \cdot 10^5$ N/m, $k_2^0 = 10^6$ N/m, $b^0 = 9800$ Ns/m, $m_1^0 = 375$ kg, $m_2^0 = 20$ kg, $k_1^-/k_1^0 = k_2^-/k_2^0 = b^-/b^0 = m_1^-/m_1^0 = m_2^-/m_2^0 = 0.95$ (empty car), $k_1^+/k_1^0 = k_2^+/k_2^0 = b^+/b^0 = m_2^+/m_2^0 = 1.05$, $m_1^+ = m_1^0 + 80$ (fully occupied car). We try to find a PD controller that minimizes overshoot of disturbance $\xi(t)$ step response (corresponding to ramp road profile) for both core and support of the plant.

For an oscillating process minimum overshoot of step disturbance response is usually achieved for phase margin around 65°. Experiments with the model indicate that acceptable performance for the support of the plant is guaranteed by the phase margin between 40° and 90° that justifies to consider target phase margin specification as $\tilde{S} = \mathrm{trap}(40°, 63°, 67°, 90°)$. All PD controllers satisfying that specification (one of them is denoted by $\circ$) are shown in $k_P - k_D$ plane in Fig. 4.15.

Let us compare that controller with two controllers obtained by other methods. The first one is PD controller determined by popular Ho's method [53] denoted by $+$ in Fig. 4.15 whereas the procedure described in [41] leads to a PD-like robust fuzzy controller denoted by $\times$ in Fig. 4.16. Comparison of disturbance step responses corresponding to ramp road profile with 10% ascent and car velocity 50 km/h plotted in Fig. 4.16 for nominal and worst-case model (only the responses with the biggest overshoot are plotted) shows significantly better performance of the proposed controller especially for the support of the plant.

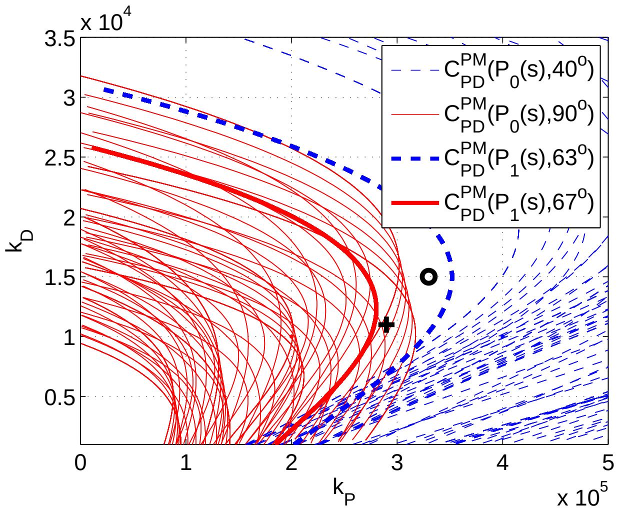

Figure 4.15: PD controllers for fuzzy linear plant with fuzzy phase margin specification – detail

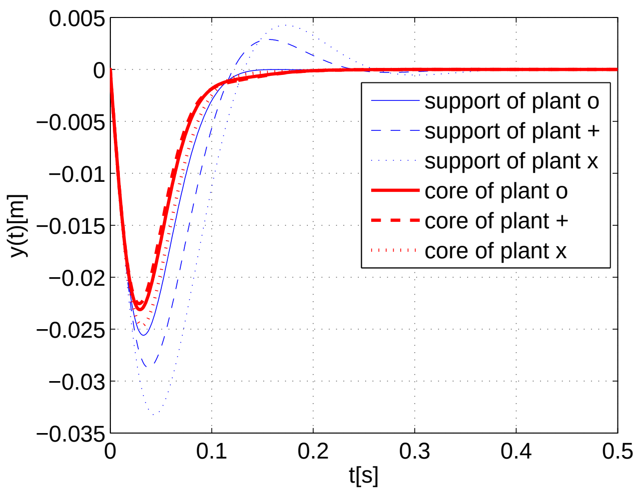

Figure 4.16: Disturbance step response of closed loop with PD controller

Chapter 5

PID controller design with respect to sensitivity margins

5.1 Introduction

Even though phase and gain margins provide a typical robustness measure and/or specification for controller design they fail for some systems since they are not reliable in the situations when gain and phase uncertainty occur at one time. It is well known that the maximum peaks of sensitivity and complementary sensitivity functions constitute more reliable performance indicator. Let us recall that sensitivity and complementary sensitivity functions for unity feedback system are defined as

$$S(s) = \frac{1}{1 + P(s)C(s)}\,, \quad T(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} \tag{5.1}$$

respectively and denote their maximum peaks as

$$M_{\mathrm{S}} = \max_{\omega} |S(\mathrm{j}\omega)|, \quad M_{\mathrm{T}} = \max_{\omega} |T(\mathrm{j}\omega)|. \tag{5.2}$$

It is usually required for M_{S} to be less than about 2 and M_{T} less than about 1.3. Larger values indicate poor performance whereas values close to 1 can lead to overdamped behaviour (that is typical for frequency response without resonance peak).

Both maximum sensitivity peaks guarantee minimum phase and gain margin. Minimum phase margin is given by

$$PM_{\min} = 2 \arcsin \frac{1}{2M_{\mathrm{S}}} \tag{5.3}$$

or

$$PM_{\min} = 2 \arcsin \frac{1}{2M_{\mathrm{T}}}\,, \tag{5.4}$$

minimum gain margin follows from

$$GM_{\min} = \frac{M_{\mathrm{S}}}{M_{\mathrm{S}} - 1}\,, \tag{5.5}$$

or

$$GM_{\min} = \frac{1}{M_{\mathrm{T}}} + 1. \tag{5.6}$$

The reverse does not hold, a value of phase or gain margin does not guarantee any bound on sensitivity peak.

Although both sensitivity margins are justified to use (sensitivity peak can be seen as robustness measure since it corresponds to the smallest distance of the open loop Nyquist plot to the critical point [-1,0j]; complementary sensitivity peak provides a reasonable approximation of total variation of reference step response) the experiments reveal that the former is more reliable.

5.2 PID controllers with respect to sensitivity peak specification

Similarly to the previous chapter let us try to depict all PI or PD controllers satisfying the maximum sensitivity or complementary sensitivity peak in $k_{\mathrm{P}} - k_{\mathrm{I}}$ or $k_{\mathrm{P}} - k_{\mathrm{D}}$ plane. Different approaches for finding a PID controller with respect to sensitivity peak specification were suggested. Åström and Hägglund [46] propose an optimization procedure, Dormido and Morilla [54] construct an additional criterion to choose a suitable controller, Garcia et al. [55] are looking for PID controller satisfying phase margin specification simultaneously.

The maximum sensitivity peak M_{S} specification is equivalent to the condition that the open loop Nyquist plot does not enter the circle with the center at $[-1, 0\mathrm{j}]$ and radius $1/M_{\mathrm{S}}$. Similarly the maximum complementary sensitivity peak M_{T} specification is equivalent to the condition that the open loop Nyquist plot does not enter the circle

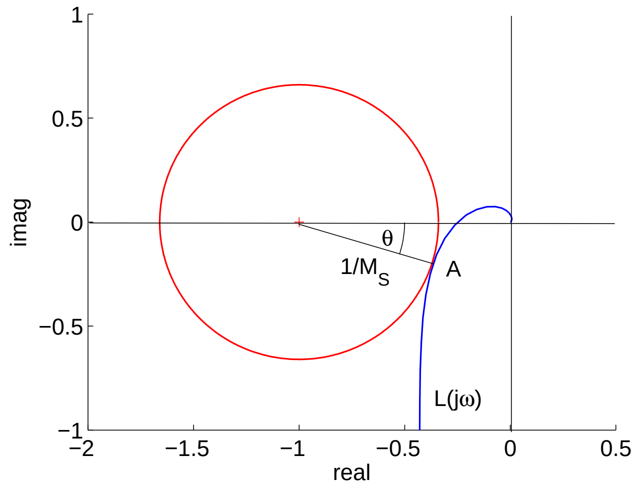

Figure 5.1: Sensitivity peak in Nyquist plot

with the center at $[-M_{\mathrm{T}}^2/(M_{\mathrm{T}}^2-1), 0\mathrm{j}]$ and radius $M_{\mathrm{T}}/(M_{\mathrm{T}}^2-1)$. The procedure will be derived only for M_{S} specification, the M_{T} case can be treated analogically.

The minimum distance of the open loop Nyquist plot $L(\mathrm{j}\omega)$ from the point $[-1, 0\mathrm{j}]$ means that the plot touches the circle in a point A (see Fig. 5.1). The condition that $L(\mathrm{j}\omega)$ goes through the point A is for a plant $P(s)$ controlled by PID controller $C(s) = k_{\mathrm{P}} + k_{\mathrm{I}}/s + k_{\mathrm{D}}s$ equivalent to

$$\left(-k_{\mathrm{P}} \cos \Phi(\omega) - \left(\frac{k_{\mathrm{I}}}{\omega} - k_{\mathrm{D}}\omega\right) \sin \Phi(\omega)\right) r(\omega) = -1 + \frac{\cos\theta}{M_{\mathrm{S}}} \tag{5.7}$$

$$\left(-k_{\mathrm{P}} \sin \Phi(\omega) + \left(\frac{k_{\mathrm{I}}}{\omega} - k_{\mathrm{D}}\omega\right) \cos \Phi(\omega)\right) r(\omega) = -\frac{\sin\theta}{M_{\mathrm{S}}} \tag{5.8}$$

whereas the tangency condition

$$\arg\left(\frac{dL(j\omega)}{d\omega}\right) = \frac{\pi}{2} - \theta; \quad \theta \in [0, \pi/2] \tag{5.9}$$

yields

$$a(\omega,\theta)k_{\mathrm{P}} + b(\omega,\theta)k_{\mathrm{I}} + c(\omega,\theta)k_{\mathrm{D}} = 0 \tag{5.10}$$

where

$$P(\mathrm{j}\omega) = r(\omega)\mathrm{e}^{\mathrm{j}(\Phi(\omega)-\pi)}$$

and

$$
\begin{aligned}
a(\omega,\theta) &= \Phi'(\omega) - \frac{r'(\omega)}{r(\omega)}\cot(\theta+\Phi(\omega)) \\
b(\omega,\theta) &= \frac{1}{\omega^2} - \frac{r'(\omega)}{\omega r(\omega)} - \frac{\Phi'(\omega)\cot(\theta+\Phi(\omega))}{\omega} \\
c(\omega,\theta) &= 1 + \frac{\omega r'(\omega)}{r(\omega)} + \omega\Phi'(\omega)\cot(\theta+\Phi(\omega))
\end{aligned}
\tag{5.11}
$$

where $r'(\omega)$ and $\Phi'(\omega)$ denote the derivatives of the magnitude and phase with respect to ω, respectively.

We face the problem of solving three nonlinear equations (5.7), (5.8) and (5.10) with five unknowns: the three PID controller parameters k_P, k_I and k_D, the frequency ω in which the specified sensitivity peak will take place and the angle θ where the contact of $L(\mathrm{j}\omega)$ with the M_S-circle will happen.

Let us choose a value of k_D, use the frequency ω as a parameter and solve the equations (5.7), (5.8) and (5.10) with respect to k_P, k_I and θ. After some algebraic manipulations and using formulas for goniometric functions we obtain the equation

$$
A(\omega,k_\mathrm{D})\sin^4\theta + B(\omega,k_\mathrm{D})\sin^3\theta + C(\omega,k_\mathrm{D})\sin^2\theta + D(\omega,k_\mathrm{D})\sin\theta + E(\omega,k_\mathrm{D}) = 0
\tag{5.12}
$$

with

$$
\begin{aligned}
A(\omega,k_\mathrm{D}) &= (r_0 r(\omega))^2 \\
B(\omega,k_\mathrm{D}) &= -\frac{r(\omega)}{M_\mathrm{S}}\Big(2\omega r(\omega)\Phi'(\omega)\cos 2\Phi(\omega) + 4k_\mathrm{D}\omega r^2(\omega)\cos\Phi(\omega) \\
&\quad + (r(\omega)-2\omega)r'(\omega)\sin 2\Phi(\omega)\Big) \\
C(\omega,k_\mathrm{D}) &= 2\omega r(\omega)r'(\omega)\left(\frac{1}{M_\mathrm{S}^2}\cos 2\Phi(\omega) - \sin^2\Phi(\omega)\right) + \omega^2 r'^2(\omega) + 4k_\mathrm{D}\omega r^2(\omega)\cdot \\
&\quad \Big(r(\omega)\sin\Phi(\omega) + \omega\Big(r(\omega)\Phi'(\omega)\cos\Phi(\omega) + k_\mathrm{D}r^2(\omega) - r'(\omega)\sin\Phi(\omega)\Big)\Big) \\
&\quad + \omega r^2(\omega)\Phi'(\omega)\Big(\omega\Phi'(\omega) + \sin 2\Phi(\omega)\Big) + r^2(\omega)\sin^2\Phi(\omega)\left(1 - \frac{2}{M_\mathrm{S}^2}\right) \\
D(\omega,k_\mathrm{D}) &= -\frac{2}{M_\mathrm{S}}\omega r(\omega)\Phi'(\omega)\Big(r(\omega)\sin^2\Phi(\omega) - \omega r'(\omega)\Big) \\
&\quad - \frac{1}{M_\mathrm{S}}\sin 2\Phi(\omega)\Big(\omega r'(\omega) - r^2(\omega)\sin^2\Phi(\omega)\Big) \\
&\quad + \frac{4}{M_\mathrm{S}}k_\mathrm{D}\omega r^2(\omega)\big(\sin^2\Phi(\omega)\cos\Phi(\omega) + \omega r'(\omega)\big)
\end{aligned}
$$

$$
\begin{aligned}
E(\omega, k_{\mathrm{D}}) \quad = \quad & -\left(\frac{1}{M_{\mathrm{S}}^2}-1\right)\left(r^2(\omega)\sin^4\Phi(\omega)-2\omega r(\omega)r'(\omega)\sin^2\Phi(\omega)+\omega^2 r'^2(\omega)\right) \\
& -4k_{\mathrm{D}}\omega r^2(\omega)\sin\Phi(\omega)\left(r^2(\omega)\sin^2\Phi(\omega)+\omega r'(\omega)+k_{\mathrm{D}}\omega r^2(\omega)\sin\Phi(\omega)\right).
\end{aligned} \tag{5.13}
$$

The equation (5.12) is solved with respect to $\sin\theta$ for a suitable range of frequencies ω. As the phase contribution of PID controller with non-negative parameters lies between $-\pi/2$ and $\pi/2$ the conditions (5.7) and (5.8) can be satisfied for $\omega_{\min} \leq \omega \leq \omega_{\max}$ with

$$
\begin{aligned}
\omega_{\min} \quad &= \quad \arg P(\mathrm{j}\omega) : \angle P(\mathrm{j}\omega) = -\frac{\pi}{2} + \arcsin\frac{1}{M_{\mathrm{S}}} \\
\omega_{\max} \quad &= \quad \arg P(\mathrm{j}\omega) : \angle P(\mathrm{j}\omega) = -\frac{3\pi}{2}\,.
\end{aligned} \tag{5.14}
$$

Naturally, satisfying the condition (5.10) makes the admissible frequency range smaller. Only the solutions of (5.12) for $\sin\theta$ lying between 0 and 1 for each ω are picked and used for computation of the pairs $(k_{\mathrm{P}}, k_{\mathrm{I}})$ that are plotted in the $k_{\mathrm{P}} - k_{\mathrm{I}}$ plane.

Simple modifications make the presented procedure usable for finding PD controllers or using complementary sensitivity peak specification.

The following example demonstrates that design of PI controller satisfying sensitivity peak may be more straightforward than using phase margin specification.

Example 5.1 – PI control with sensitivity peak specifications

Let us consider the plant

$$
P(s) = \frac{-s+2}{(s+1)^3} \tag{5.15}
$$

and find a suitable PI controller based on phase margin and sensitivity peak specifications.

In Fig. 5.2 PI controllers for phase margin $45°, 60°$ and $75°$ are plotted. To choose a suitable controller from each boundary we use the generally accepted rule that for sufficient damping (reasonably high phase margin) the controllers with maximum integral part (denoted by $\times$) resulting for instance in minimum integral error produce satisfactory response, see [50]. The corresponding step responses are depicted in Fig. 5.3.

One can see that neither of the responses is satisfactory. This is due to the fact that the slope of the open loop around the crossover frequency is too low. To increase the negative slope we should use PI controller with higher break point frequency $\omega_{\mathrm{I}} = k_{\mathrm{I}}/k_{\mathrm{P}}$

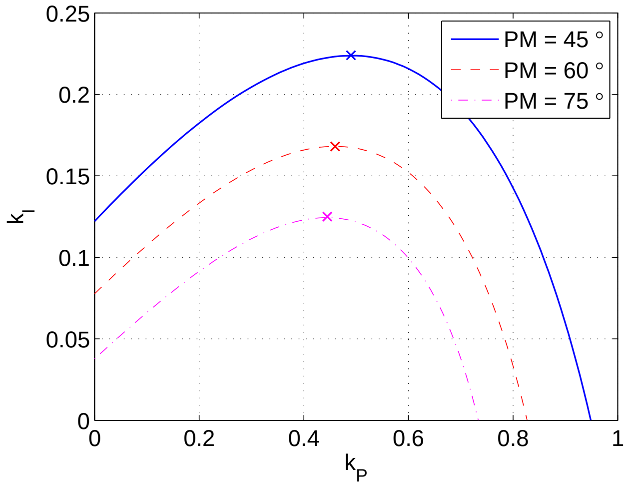

Figure 5.2: PI controllers with phase margin specification

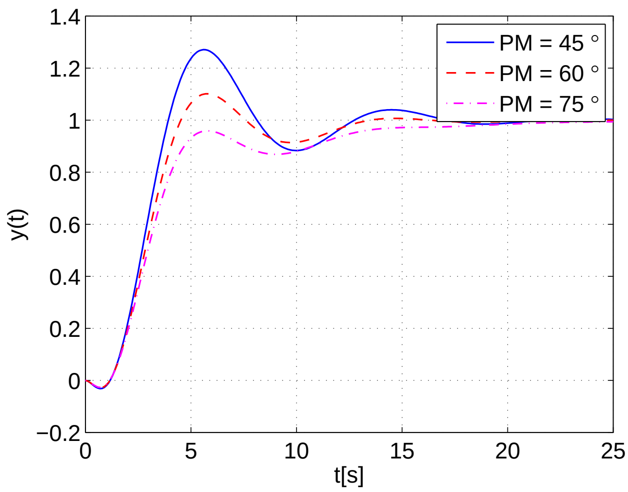

Figure 5.3: Step responses of PI controllers with phase margin specification

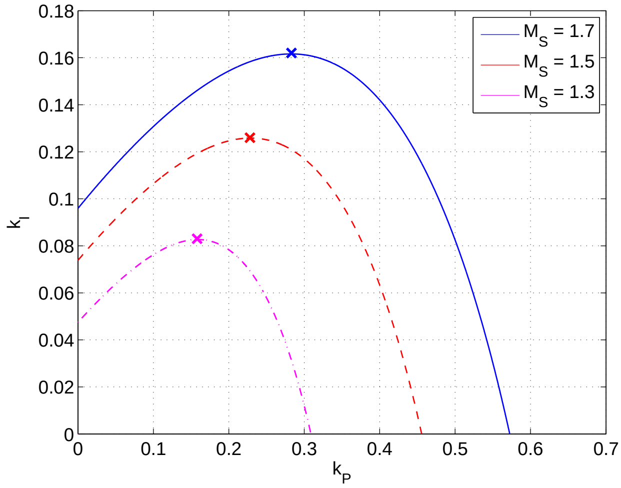

Figure 5.4: PI controllers with sensitivity peak specification

that enlarges the frequency range where integral action applies. Therefore to find a better controller we need to move leftwards on each curve in the $k_P - k_I$ plane.

Let us apply the procedure described above and plot all PI controllers satisfying the sensitivity peak specification. The result for $M_S = 1.3, 1.5$ and 1.7 is shown in Fig. 5.4.

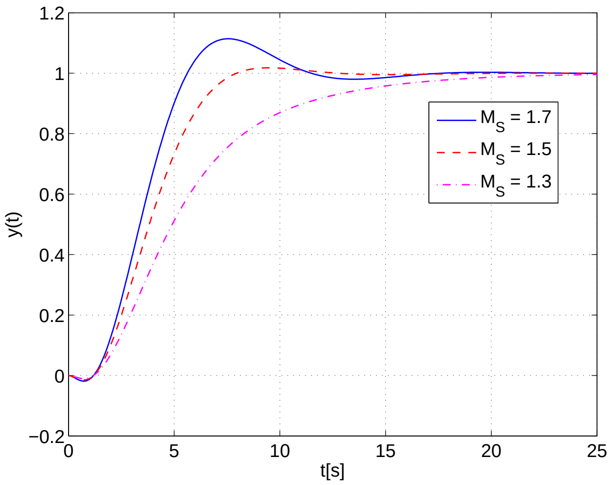

Figure 5.5: Step responses of PI controllers with sensitivity peak specification

Let us again choose the controllers with maximum integral part. Each of the corresponding step responses depicted in Fig. 5.5 is satisfactory, the value of sensitivity peak can be used as a tuning parameter for admissible step response overshoot.

Naturally, as all the results derived in the chapter 4 for phase margin applies for sensitivity peak as well, one can use similar procedure to find PI or PD controller with fuzzy target set of specifications given by sensitivity peak.

Chapter 6

Conclusion

The thesis deals with several problems concerning analysis and synthesis of continuous-time linear systems with fuzzy parametric uncertainty. In chapter 2 an algorithm for determining minimum confidence level preserving stability of those systems is presented. Firstly a system with independent uncertainty structure is considered. Next, the algorithm is generalized for the case when the coefficients of characteristic polynomial are supposed to be characterized by a multivariate non-symmetric ellipsoidal membership function. In both cases arbitrary shape of membership functions is supposed. The algorithm is graphical in nature and is based on generalization of Tsypkin-Polyak plot.

In chapter 3 a more realistic case is considered when the coefficients of the characteristic polynomial are linear affine functions of plant parameters described by fuzzy numbers rather than being the coefficients themselves. This is for instance the case when a plant with the transfer function coefficients described by fuzzy numbers is controlled with a fixed controller. In contrast to classical robust control approach, since arbitrarily shaped nonsymmetric membership functions are considered, the presented approach can deal with the systems whose operating point does not lie in the middle of admissible parameter intervals that represents a common situation.

The chapter 4 addresses the problem of PI and PD controller design for linear plants with fuzzy parametric uncertainty. The design is understood as achieving an inclusion of fuzzy set of plants onto fuzzy set of target phase or gain margin specifications. Such formulation allows to degrade closed loop specifications that can result in more acceptable performance than in the cases where the controller is designed for most-cases model of the plant regardless of a model in worst-case conditions and vice versa that is demonstrated on a real model and a simulation example.

Chapter 5 is concerned with PI and PD controller design under sensitivity and complementary sensitivity peaks specifications. Those specifications are proved to be more reliable tuning parameters than usually used phase and gain margins that is illustrated on a simple example. The proposed procedure depicts all the controllers satisfying sensitivity margins in $k_\mathrm{P} - k_\mathrm{I}$ or $k_\mathrm{P} - k_\mathrm{D}$ plane that allows to combine it with other criteria as maximum integration part or specified frequency band.

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